Eur. Phys. J. C 25, 259–276 (2002)
Digital Object Identifier (DOI) 10.1007/s10052-002-0975-1

THE EUROPEAN
PHYSICAL JOURNAL C

Analysis of the pion–kaon sigma term and related topics*

M. Frink[a], B. Kubis[b], U.-G. Meißner[c]

Forschungszentrum Jülich, Institut für Kernphysik (Theorie), 52425 Jülich, Germany

Received: 21 March 2002 /
Published online: 5 July 2002 –

Abstract. We calculate the one-loop contributions to the difference $\Delta_{\pi K}$ between the isoscalar on-shell pion–kaon scattering amplitude at the Cheng–Dashen point and the scalar form factor $\Gamma_K(2M_\pi^2)$ in the framework of three flavor chiral perturbation theory. These corrections turn out to be small. This is further sharpened by treating the kaons as heavy fields (two flavor chiral perturbation theory). We also analyze the two-loop corrections to the kaon scalar form factor based on a dispersive technique. We find that these corrections are smaller than in the comparable case of the scalar form factor of the pion. This is related to the weaker final state interactions in the pion–kaon channel.

1 Introduction

In QCD, the mass terms for the three light quarks, u, d and s, can be measured in the so-called sigma terms. These are matrix elements of the scalar quark currents $m_q \bar{q} q$ in a given hadron H, $\langle H|m_q\bar{q}q|H\rangle$, with H e.g. pions, kaons or nucleons. Since no external scalar probes are available, the determination of these matrix elements proceeds by analyzing four-point functions, more precisely Goldstone boson–hadron scattering amplitudes in the unphysical region, $\phi(q)+H(p) \to \phi(q')+H(p')$ (note that the hadron can also be a Goldstone boson). The determination of the sigma terms starts from the generic low-energy theorem (such a low-energy theorem was first formulated for pion–nucleon scattering [1]) for the isoscalar scattering amplitude $A(\nu,t)$:

$$F^2 A(t,\nu) = \Gamma(t) + q'^\mu q^\nu r_{\mu\nu}, \qquad (1.1)$$

where F is the Goldstone boson decay constant and $\Gamma(t)$ is the pertinent scalar form factor

$$\Gamma(t) = \langle H(p')|m_q\bar{q}q|H(p)\rangle, \quad t=(p'-p)^2, \qquad (1.2)$$

employing the standard Mandelstam variables s,t,u to describe the scattering process, with $s+t+u = 2M_H^2 + 2M_\phi^2$, and further introducing the crossing variable $\nu = s-u$. At zero momentum transfer, this scalar form factor gives the desired sigma term,

$$\Gamma(0) = 2M_H\sigma_{\phi H}, \qquad (1.3)$$

for appropriately normalized hadron states (note that sometimes one uses M_ϕ for the normalization). Furthermore, in (1.1) $r_{\mu\nu}$ is the so-called remainder, which is not determined by chiral symmetry. However, it has the same analytical structure as the scattering amplitude. To determine the sigma term, one has to work in a kinematic region where this remainder is small, otherwise a precise determination is not possible. Beyond tree level, the region where the remainder is small shrinks to the so-called Cheng-Dashen (CD) point [2], which e.g. for pion scattering off other hadrons is given by

$$t = 2M_\pi^2, \quad \nu = 0, \qquad (1.4)$$

which clearly lies outside the physical region for elastic scattering but well inside the Lehmann ellipse. The reaction most studied to determine a sigma term is certainly elastic pion–nucleon scattering, $\pi N \to \pi N$, but the best understood process is low energy pion-pion scattering $\pi\pi \to \pi\pi$ (for a beautiful sigma term analysis for that case, see [3]). Much less is known for processes involving kaons, in particular for (anti)kaon–nucleon scattering, which is of interest for particle, nuclear and astrophysics. One of the reasons is the large kaon mass, which moves the corresponding CD-point to $t = 2M_K^2$, far away from the physical region. That makes the interpolation from the data much more difficult than in the pion case. In addition, there are open channels below threshold or even resonances (for $\bar{K}N \to \bar{K}N$). There are also less high precision scattering data. Before addressing these issues, it is therefore mandatory to understand the simplest process involving strange quarks, i.e. elastic pion–kaon scattering. This reaction has attracted much recent interest (see e.g. [4]) mostly triggered by the intended lifetime measurement of πK atoms at CERN [5], but also as a theoretical laboratory to study the question whether the kaon can be considered as a heavy particle, see [6,7]. Therefore, as an

* Work supported in part by Studienstiftung des deutschen Volkes

[a] e-mail: m.frink@fz-juelich.de
[b] e-mail: b.kubis@fz-juelich.de
[c] e-mail: u.meissner@fz-juelich.de

intermediate step it was proposed to analyze the sigma term in pion–kaon scattering [3]. This is done in this paper in two ways. In Sect. 3 we use standard three flavor chiral perturbation theory (CHPT) [8], treating the pions and the kaons as (pseudo-) Goldstone bosons of the spontaneously broken chiral symmetry of QCD. We discuss the one-loop representation of the scalar pion–kaon form factor and of the isospin-even πK scattering amplitude and deduce the size of the remainder at the CD-point. In Sect. 4 we analyze the sigma term in the heavy-kaon framework, which helps us to understand the results obtained in SU(3) CHPT. To further analyze the stability of our results, we calculate in Sect. 5 the two-loop corrections to the scalar pion–kaon form factor in the threshold region. We employ the dispersive technique of [9], despite the fact that the technology for explicit two-loop corrections exists (see e.g. [10]). However, for the estimate of these effects as needed here the use of analyticity and unitarity combined with chiral constraints is sufficient. We end this paper with a short summary in Sect. 6. A related low-energy theorem for soft kaons is analyzed in Appendix A. Further technicalities and definitions are relegated to the following appendices.

2 Basic considerations

Elastic pion–kaon scattering can be parameterized in terms of an isospin 1/2 and an isospin 3/2 amplitude, called $T^{1/2}(s,t)$ and $T^{3/2}(s,t)$, respectively, and s,t,u are the conventional Mandelstam variables. Note that since these are subject to the constraint $s+t+u=2M_\pi^2+2M_K^2$, it suffices to specify two arguments, like s and t or ν and t. Here, we are interested in the isoscalar amplitude

$$T^+_{\pi K}(s,t) \equiv A_{\pi K}(s,t) = \frac{1}{3}T^{1/2}(s,t) + \frac{2}{3}T^{3/2}(s,t). \quad (2.1)$$

More precisely, this amplitude can be obtained entirely from the isospin 3/2 amplitude because of the $s \leftrightarrow u$ crossing relation (for clarity, we exhibit here all three arguments of the scattering amplitude),

$$T^{1/2}(s,t,u) = \frac{3}{2}T^{3/2}(u,t,s) - \frac{1}{2}T^{3/2}(s,t,u). \quad (2.2)$$

Furthermore, the reaction $\pi^+(q) + K^+(p) \to \pi^+(q') + K^+(p')$ defines the isospin 3/2 amplitude,

$$\begin{aligned}&\langle \pi^+(q')K^+(p')\text{out}|\pi^+(q)K^+(p)\text{in}\rangle \\ &= \mathrm{i}(2\pi)^4\delta^{(4)}(p+q-p'-q')T^{3/2}(s,t,u). \quad (2.3)\end{aligned}$$

Note also that T^+ is even under $s \leftrightarrow u$ crossing, while the isovector amplitude $T^- = (T^{1/2}-T^{3/2})/3$ is odd. The partial wave expansion for the πK scattering amplitudes takes the form

$$T^I(s,t) = 32\pi\sum_{l=0}^{\infty}(2l+1)t_l^I(q)P_l(z), \quad (2.4)$$

in terms of the squared momentum transfer $t=-2q^2(1-z)$ and the cosine of the scattering angle, $z=\cos(\theta)$.

The low-energy theorem, (1.1), takes the form

$$F^2 A^{\rm CD}_{\pi K} = \Gamma_K(2M_\pi^2) + \Delta_{\pi K}. \quad (2.5)$$

with F^2 expressed in terms of the pion (F_π) or the kaon (F_K) decay constants or the product thereof. From the view point of the chiral expansion, all of these choices are legitimate. This has most notable consequences for the remainder because it affects its leading (fourth) order expression. Therefore, the fact that $F_\pi \neq F_K$ will play an important role in the numerical analysis discussed below, related to a particular chiral SU(2) breaking effect within a three flavor calculation (as explained below). The pertinent scalar kaon form factor is

$$\Gamma_K(t) = \langle K^0(p')|\hat{m}(\bar{u}u+\bar{d}d)|K^0(p)\rangle, \quad \hat{m} = \frac{1}{2}(m_u+m_d). \quad (2.6)$$

At $t=0$ this defines the πK sigma term,

$$2M_\pi\sigma_{\pi K} = \Gamma_K(0). \quad (2.7)$$

In what follows, we will analyze the size of the remainder at the CD-point in the isospin limit $m_u=m_d=\hat{m}$ to one loop accuracy, neglecting also electromagnetic isospin violation. To get an idea about possible higher order corrections, we will also calculate the scalar form factor $\Gamma_K(t)$ beyond one loop, following the approach of [9]. In Appendix A, we analyze a similar low-energy theorem taking the kaons as soft.

3 Analysis of $\sigma_{\pi K}$ in SU(3) chiral perturbation theory

The tool to systematically calculate low-energy QCD Green functions and transition currents is chiral perturbation theory. This amounts to a systematic expansion around the chiral limit in terms of two small parameters related to the quark masses and the external momenta [8]. In the chiral limit of vanishing quark masses, pions, kaons and etas are massless Goldstone bosons, but in nature the quark masses are finite, in particular the strange quark is much heavier than the light up and down quarks, which is reflected in the difference of the expansion parameters for two and three flavor CHPT, $M_\pi^2/(4\pi F_\pi)^2 = 0.02$ and $M_K^2/(4\pi F_\pi)^2 = 0.2$, respectively. This large difference is at the heart of the heavy kaon approach to be discussed below. Here, we analyze the πK sigma term to the first non-trivial order, i.e. to one loop accuracy in the standard scenario of a large quark condensate, based on the one-loop representation for πK scattering given in [11]. An analysis of πK scattering to leading order in generalized CHPT can be found in [12]. Also needed in the analysis of the remainder at the CD-point is the fourth order representation of the πK scalar form factor, first given explicitly in [13]. It has the form

$$\Gamma_K(t) = \frac{M_\pi^2}{2}\left\{1 + \frac{1}{F^2}\left[L_4^r(-32M_K^2+16t)\right.\right.$$

$$
\begin{aligned}
&+ L_5^r(8M_\pi^2 - 16M_K^2 + 4t) + L_6^r 64 M_K^2 \\
&+ L_8^r(-16M_\pi^2 + 32M_K^2) \\
&- \frac{1}{2}M_\pi^2\mu_\pi + \left(-\frac{1}{6}M_\pi^2 + \frac{2}{3}M_K^2\right)\mu_\eta - \frac{3}{4}tJ_{\pi\pi}^r(t) \\
&- \frac{3}{4}tJ_{KK}^r(t) + \left(\frac{2}{9}M_K^2 - \frac{1}{4}t\right)J_{\eta\eta}^r(t)\Bigg]\Bigg\} + \mathcal{O}(p^6). \quad (3.1)
\end{aligned}
$$

Here, the J_{PQ}^r are the renormalized loop functions as defined in [8] and λ is the scale of dimensional regularization. We set $\lambda = M_\rho$. We use the operator basis of [8]. Furthermore,

$$
\mu_P = \frac{1}{(4\pi)^2}\ln\frac{M_P^2}{\lambda^2} \quad (P = \pi, K, \eta). \tag{3.2}
$$

We remark that setting $F = F_\pi$, the fourth order contribution amounts to a 22% correction to the tree level result at the two-pion threshold, $t = 4M_\pi^2$, which is fairly small for a three flavor observable. For comparison, the scalar form factor of the pion is affected by a 29% correction at the two-pion threshold. We will come back to this topic in Sect. 5. It is also of interest to analyze the Taylor expansion of $\Gamma_K(t)$ around $t = 0$,

$$
\Gamma_K(t) = \Gamma_K(0)\left(1 + \frac{1}{6}\langle r_S^2\rangle_K t + \mathcal{O}(t^2)\right), \tag{3.3}
$$

in terms of the scalar radius. To get a handle on the theoretical uncertainty, we use two sets of values for the low-energy constants L_i and their corresponding uncertainties. Set 1 is from [14] and set 2 from [15] (more precisely, we use the so-called central fit). We find

$$
\langle r_S^2\rangle_K = \begin{cases} (0.30 \pm 0.23)\,\mathrm{fm}^2, & \text{set 1}, \\ (0.38 \pm 0.02)\,\mathrm{fm}^2, & \text{set 2}. \end{cases} \tag{3.4}
$$

Two remarks are in order. First, the central value is smaller than the scalar pion radius, $\langle r_S^2\rangle_\pi \simeq 0.6\,\mathrm{fm}^2$, pointing towards smaller final state interactions. Second, the uncertainty due to the LECs is fairly large for set 1 but much smaller for set 2. This can be traced back to the circumstance that in the second set, the variation in the OZI-violating LEC L_4 is set to zero, $\Delta L_4 = 0$, but it is sizable for the first set, $\Delta L_4 \simeq 0.5 \cdot 10^{-3}$. This shows that this observable is very sensitive to this particular LEC. The large uncertainty due to the variations in the LEC also indicates that the chiral logarithms encoded in the loop contribution play a less distinct role as compared to isoscalar S-wave pion–pion interactions. We note that the normalization of the form factor is $\Gamma_K(0) = 0.52(0.53)M_\pi^2$ for set 1 (2).

We turn to the remainder at the CD-point. To leading order (tree level) it vanishes, as noted before. From the explicit one-loop expressions for the scalar form factor and for the πK scattering amplitude, it is straightforward to deduce the expression for the remainder at the CD-point. For completeness, we give here its explicit form using the normalization $F^2 = F_\pi^2$, which is natural if one considers the kaon as the heavy particle (much like a nucleon) from which the pion scatters. We obtain[1]

$$
\begin{aligned}
\Delta_{\pi K}^{\mathrm{CD}} = \frac{M_\pi^4}{F^2}\Bigg[&L_{\mathrm{CD}}^r(\lambda) + \sum_{P=\pi,K,\eta}\frac{\mathcal{P}_P}{(4\pi)^2}\ln\frac{M_P}{\lambda} \\
&+ \mathcal{P}_1 J_{\pi K}^r(M_K^2) + \mathcal{P}_2 J_{K\eta}^r(M_K^2) - \frac{\mathcal{P}_3}{(4\pi)^2}\Bigg], \quad (3.5)
\end{aligned}
$$

with $L_{\mathrm{CD}}^r(\lambda) = 2(4L_2^r(\lambda) + L_3 - 2L_5^r(\lambda) + 4L_8^r(\lambda))$ a combination of low-energy constants. Furthermore,

$$
\begin{aligned}
&\mathcal{P}_\pi = x(x-1), \quad \mathcal{P}_K = \frac{1}{6}(1-2x), \\
&\mathcal{P}_\eta = \frac{1}{9}\left(x^2 - x - 2\right), \quad \mathcal{P}_1 = \frac{1}{2}\left(-x^2 + \frac{3x}{2} - 1\right), \\
&\mathcal{P}_2 = \frac{1}{9}\left(1 - \frac{x}{4} - \frac{x^2}{2}\right), \quad \mathcal{P}_3 = \frac{1}{6}\left(1 + \frac{x}{2}\right), \quad (3.6)
\end{aligned}
$$

with $x = M_\pi^2/(2M_K^2) \simeq 1/26$. It is remarkable that no terms $\propto M_\pi^2 M_K^2$ appear. This is, of course, different if one chooses $F^2 = F_\pi F_K$ or $F^2 = F_K^2$ because

$$
\frac{F_K}{F_\pi} = 1 + \frac{4L_5^r}{F_0^2}(M_K^2 - M_\pi^2) + \text{chiral logs}, \tag{3.7}
$$

where F_0 is the leading term in the quark mass expansion of the Goldstone boson decay constants. This will reflect itself in terms $\sim L_5^r M_\pi^2 M_K^2$ in $\Delta_{\pi K}^{\mathrm{CD}}$. We refrain, however, from giving the complete analytical formulae for these cases here.

Our numerical results for the amplitude, the scalar form factor and the relative size of the remainder at the CD-point, $R = \Delta_{\pi K}/(F^2 A_{\pi K}^{\mathrm{CD}})$, are collected in Table 1. For the choice $F = F_\pi^2$, the remainder at the CD-point is very small, constituting a true SU(2) result, as explained below. Even for the choice of $F^2 = F_\pi F_K$, the resulting numbers are still on the low side expected from SU(3) breaking $\sim (M_K/\Lambda_\chi)^2 \simeq 0.2$ (with $\Lambda_\chi = 4\pi F_\pi$). This observation also holds individually for the scattering amplitude and for the form factor at the CD-point. These results are comparable to what is found in the analysis of the pion sigma term [3]. If one normalizes $F^2 A_{\pi K}^{\mathrm{CD}}$ and $\Gamma_{\pi K}(2M_\pi^2)$ at tree level to one (for an easier comparison with the pion case, see below), the first row of Table 1 reads

$$
\begin{aligned}
1.14 \quad &= 1.10 \quad + 0.04\,, \\
F^2 A_{\pi K}^{\mathrm{CD}} &= \Gamma_K(2M_\pi^2) + \Delta_{\pi K},
\end{aligned} \tag{3.8}
$$

astonishingly close to the tree level result. The remainder amounts to a correction of $0.04M_\pi/2 \simeq 2.8\,\mathrm{MeV}$. It is instructive to give the results for the pion case [3]

$$
\begin{aligned}
1.14 \quad &= 1.09 \quad + 0.05 \\
F_\pi^2 A_\pi^{\mathrm{CD}} &= \Gamma_\pi(2M_\pi^2) + \Delta_\pi.
\end{aligned} \tag{3.9}
$$

We note that the remainder is comparable to the pion–kaon case; for the pion sigma term it amounts to a correction of about 3.5 MeV.

[1] Note that in terms of order p^4, we always set $F = F_\pi$. This is legitimate to the accuracy we are working

Table 1. Size of the remainder at the CD-point for various choices of the meson decay constants and the low-energy constants $L_i^r(M_\rho)$

F^2	L_i set	$F^2 A_{\pi K}^{\mathrm{CD}}$ $[M_\pi^2]$	$\Gamma_K(2M_\pi^2)$ $[M_\pi^2]$	R [%]
F_π^2	1	0.572	0.551	3.7
$F_\pi F_K$	1	0.679	0.551	18.9
F_K^2	1	0.785	0.551	29.8
F_π^2	2	0.600	0.587	2.2
$F_\pi F_K$	2	0.681	0.587	13.8
F_K^2	2	0.762	0.587	23.0

To further illustrate the situation we consider so-called scale relations. This amounts to an expansion of both the scalar form factor and the πK amplitude at the CD-point in powers of M_π and representing the occurring terms in terms of the chiral logarithms $\ln(M_P^2/\Lambda^2)$, $P \in \{\pi, K, \eta\}$, via appropriately defined scales Λ_i. The results are only given for the $1/F_\pi^2$ choice of normalization. Consider first Γ_K at the CD-point. The non-logarithmic terms in the coefficients of M_π^4 respectively $M_\pi^2 M_K^2$, i.e. constant terms and the contributions proportional to the LECs, are absorbed into common scales $\Lambda_{1/2}$. The scales so defined, $\Lambda_{1/2}$, are unique and independent of the meson masses. We find

$$\begin{aligned}\Gamma_K(2M_\pi^2) = {} & \frac{M_\pi^2}{2} \\ & + \frac{M_\pi^2}{(4\pi F)^2}\Bigg[M_\pi^2\left(-\ln\left(\frac{M_\pi^2}{\Lambda_1^2}\right) - \frac{3}{4}\ln\left(\frac{M_K^2}{\Lambda_1^2}\right)\right. \\ & \left. - \frac{1}{3}\ln\left(\frac{M_\eta^2}{\Lambda_1^2}\right)\right) + \frac{4}{9}M_K^2\ln\left(\frac{M_\eta^2}{\Lambda_2^2}\right) \\ & + \frac{217}{720}\frac{M_\pi^4}{M_K^2} + \frac{1417}{20160}\frac{M_\pi^6}{M_K^4}\Bigg] + \mathcal{O}\left(\frac{M_\pi^{10}}{F^2 M_K^6}\right),\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}\Lambda_1 = {} & \\ & \lambda\exp\left[\frac{6}{25}\left((4\pi)^2(16L_4^r + 8L_5^r - 8L_8^r) - \frac{3}{8}\pi - \frac{5}{18}\right)\right] \\ & = 527\,\mathrm{MeV}, \\ \Lambda_2 = {} & \\ & \lambda\exp\left[\frac{9}{8}\left((4\pi)^2(16L_4^r + 8L_5^r - 32L_6^r - 16L_8^r) - \frac{1}{9}\right)\right] \\ & = 511\,\mathrm{MeV},\end{aligned} \tag{3.11}$$

employing the LEC values of set 1. Note that these expressions are independent of the regularization scale λ since the logarithmic scale dependence of the L_i cancels the explicit factor of λ. The corresponding expression for $F_\pi^2 A_{\pi K}^{\mathrm{CD}}$ reads

$$\begin{aligned}F_\pi^2 A_{\pi K}^{\mathrm{CD}} = {} & \frac{M_\pi^2}{2} \\ & + \frac{M_\pi^2}{(4\pi F)^2}\Bigg[M_\pi^2\left(-\ln\left(\frac{M_\pi^2}{\tilde\Lambda_1^2}\right) - \frac{9}{8}\ln\left(\frac{M_K^2}{\tilde\Lambda_1^2}\right)\right. \\ & \left. - \frac{3}{8}\ln\left(\frac{M_\eta^2}{\tilde\Lambda_1^2}\right)\right) + \frac{4}{9}M_K^2\ln\left(\frac{M_\eta^2}{\tilde\Lambda_2^2}\right) - \frac{\pi}{2}\frac{M_\pi^3}{M_K} \\ & + \frac{M_\pi^4}{M_K^2}\left(-\frac{1}{2}\ln\left(\frac{M_\pi^2}{\lambda^2}\right) + \frac{35}{64}\ln\left(\frac{M_K^2}{\lambda^2}\right) - \frac{3}{64}\ln\left(\frac{M_\eta^2}{\lambda^2}\right)\right. \\ & \left. + \frac{901}{2160} - \frac{1}{36}\sqrt{2}\arctan(\sqrt{2})\right) + \frac{7\pi}{16}\frac{M_\pi^5}{M_K^3} \\ & + \frac{M_\pi^6}{M_K^4}\left(\frac{5}{16}\ln\left(\frac{M_\pi^2}{\lambda^2}\right) - \frac{81}{256}\ln\left(\frac{M_K^2}{\lambda^2}\right) + \frac{1}{256}\ln\left(\frac{M_\eta^2}{\lambda^2}\right)\right. \\ & \left. - \frac{3181}{15120} - \frac{23}{1152}\sqrt{2}\arctan(\sqrt{2})\right)\Bigg] \\ & + \mathcal{O}\left(\frac{M_\pi^{10}}{F^2 M_K^6}\right),\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\tilde\Lambda_1 &= \lambda\exp\left[\frac{1}{5}\left((4\pi)^2(8L_2^r + 2L_3^r + 16L_4^r + 4L_5^r)\right.\right. \\ & \qquad \left.\left. - \frac{3}{8}\pi - \frac{1}{18} + \frac{4}{27}\sqrt{2}\arctan(\sqrt{2})\right)\right] = 724\,\mathrm{MeV}, \\ \tilde\Lambda_2 &= \lambda\exp\left[\frac{9}{8}\left((4\pi)^2(16L_4^r + 8L_5^r - 32L_6^r - 16L_8^r)\right.\right. \\ & \qquad \left.\left. - \frac{1}{9}\right)\right] = 511\,\mathrm{MeV}.\end{aligned} \tag{3.13}$$

For the terms of order $\mathcal{O}(M_\pi^6/F^2M_K^2)$ and $\mathcal{O}(M_\pi^8/F^2M_K^4)$ it is not possible to incorporate the constants in the coefficients into a universal scale in each of the chiral logarithms, as can be traced back to their origin in the regularization procedure. Chiral logarithms never appear alone; their occurrence is, moreover, accompanied by a pole term L. In the absence of counterterms proportional to $M_\pi^6/(F^2M_K^2)$ or $M_\pi^8/(F^2M_K^4)$, renormalizability requires the coefficients of L to add up to zero for a given order, and the same is therefore true for the coefficients of the logarithms. It is thus impossible to allocate to each chiral logarithm its proportionate fraction of the non-logarithmic contributions, and neither is it then possible to define a common scale. For the purpose of illustrating the similarity structure of the scalar form factor and of the scattering amplitude we analyze the scales in those terms which behave as M_π^4 and $M_\pi^2 M_K^2$. With Λ_2 and $\tilde\Lambda_2$ close to the eta mass it is clear that the potentially large corrections involving M_K^2 are individually small. Since both scales are even identical, these contributions cancel completely in the remainder. In the case of Λ_1 and $\tilde\Lambda_1$ cancellations are not complete, as $\tilde\Lambda_1$ is larger than Λ_1, whose value is again not much different from the eta mass.

4 Analysis of $\sigma_{\pi K}$ in the heavy-kaon approach

So far, we have considered the kaons and the pions on equal footing, namely as pseudo-Goldstone bosons of the spontaneously broken chiral symmetry of QCD, with their finite masses related to the non-vanishing current quark masses. However, the fact that the kaons (and also the eta) are much heavier than the pions might raise the question whether a perturbative treatment in the strange quark mass is justified. In fact, one can take a very different view and consider only the pions as light with the kaons behaving as heavy sources, much like a conventional matter field in baryon CHPT. This point of view was first considered in the Skyrme model [16] and has been reformulated in the context of heavy-kaon chiral perturbation theory (HKCHPT) in [6] (a closely related work applying reparameterization invariance instead of the reduction of relativistic amplitudes was presented in [7].). Since the kaons appear now as matter fields, the chiral Lagrangian for the pion–kaon interaction decomposes into a string of terms with a fixed number of kaon fields, that is, into sectors with n ($n \geq 0$) in-coming and n out-going kaons. Here, we consider processes with at most one kaon in the in/out states. Obviously, the power counting has to be modified due to the new large mass scale, M_K, and as is the case for baryons, terms with an odd number of derivatives are allowed. For keeping the paper self-contained, we give in Appendix B a more detailed discussion of the heavy-kaon formulation, following essentially [6]. This approach is particularly suited to analyze chiral SU(2) theorems for three flavor observables, and it is therefore natural to reconsider pion–kaon scattering and the issues related to it discussed in the previous sections[2]. The heavy kaon formulation can be connected to the standard SU(3) CHPT approach by so-called matching relations, which will be discussed in some detail below for the case of the scalar form factor Γ_K. In general, this need not be done, but for practical purposes it cannot be avoided; there are simply not enough precise low-energy data to pin down the heavy kaon LECs independently. The corresponding heavy-kaon Lagrangian for doing this matching is displayed in Appendix C.

Let us first consider the scalar form factor of the kaon. The calculation proceeds as in the standard case, only we now have to consider new vertices and the loops are entirely pionic ones. We will again work with the physical masses and employ dimensional regularization. To one-loop accuracy, one finds the following renormalized (finite) representation for Γ_K:

$$
\begin{aligned}
\Gamma_K(t) = \frac{M_\pi^2}{2}\Bigg[8\Bigg(& (A_3^r + 2A_4^r) \\
& \times \left(-\frac{1}{2M_\pi^2} + 2C_5^r + 4C_6^r + \frac{1}{F^2} l_3^r \right) - 4C_{13}^r \\
& - 4C_{14}^r - 8C_{15}^r - \frac{1}{8(4\pi F)^2} A_2^r M_K^2 \Bigg) M_\pi^2 \\
& + \left(2C_5^r + 4C_6^r + \frac{1}{(4\pi F)^2} \frac{A_2^r}{6} M_K^2 \right) t \\
& + \frac{1}{F^2}\left(8A_3^r + 16A_4^r + 3A_1^r + \frac{A_2^r}{2} M_K^2 \right) M_\pi^2 \mu_\pi \\
& + \frac{1}{F^2}\Bigg((3A_1^r + A_2^r M_K^2 + 6A_3^r + 12A_4^r) M_\pi^2 \\
& + \left(-\frac{3}{2}A_1^r - \frac{1}{4}A_2^r M_K^2 \right) t \Bigg) J_{\pi\pi}^r(t) \Bigg] + \mathcal{O}(p^6). \quad (4.1)
\end{aligned}
$$

In this and in all following formulae, one has $F = F_\pi$. To provide HKCHPT with predictive power we need the numerical values of the renormalized LECs characteristic of the heavy-kaon theory, more specifically the A_i^r and C_i^r appearing in (4.1). In principle, these can be obtained from experimental data, in complete analogy to the determination of the L_i in conventional SU(3) CHPT. In fact, one can translate knowledge of the L_i into the heavy-kaon theory and thus infer information about the HKCHPT parameters. The major difference in both approaches is the treatment of the strange quark mass. While in SU(3) CHPT m_s serves as an expansion parameter of the chiral series, in the heavy-kaon approach m_s does enter as part of the static kaon mass; yet, when involved in loops, the kaon is rather dealt with as a heavy quark, i.e. its effects are absorbed into the numerical values of the constants present in any expansion. Having calculated an observable quantity in both schemes, a comparison of the two power series then yields an expansion of certain combinations of HKCHPT parameters in powers of m_s, where the SU(3) CHPT LECs are incorporated into the coefficients. One can thus carry out an order by order investigation as to the role which the strange quark mass plays for the heavy-kaon constants, and as for their dependence on the renormalization scale. This procedure is referred to as matching. The matching relations of a sufficient number of observables then provide enough relations among the parameters to solve for each of them separately. We now return to the scalar form factor of the kaon. For the matching procedure, we have to bring Γ_K from (3.1) into a form which allows for a direct comparison with its heavy-kaon counterpart, (4.1). We rewrite the form factor in terms of $\bar{M}_K$ (see Appendix B for definition of this and related quantities) and expand in powers of the light quark mass $\hat{m}$ and the squared momentum transfer t. Using (B.11), we find

$$
\begin{aligned}
\Gamma_K(t) = & \frac{M_\pi^2}{2} \\
& + \frac{M_\pi^2}{2F^2}\Bigg[\bar{M}_K^2 \Bigg(-32L_4^r - 16L_5^r + 64L_6^r + 32L_8^r \\
& + \frac{2}{9}\frac{1}{(4\pi)^2} + \frac{8}{9}\frac{1}{(4\pi)^2} \ln\left(\frac{4}{3}\frac{\bar{M}_K^2}{\lambda^2} \right) \Bigg) \\
& + M_\pi^2 \Bigg(-16L_4^r + 32L_6^r + \frac{1}{3}\frac{1}{(4\pi)^2} \\
& + \frac{5}{18}\frac{1}{(4\pi)^2} \ln\left(\frac{4}{3}\frac{\bar{M}_K^2}{\lambda^2} \right) \Bigg) - \frac{1}{2} M_\pi^2 \mu_\pi
\end{aligned}
$$

[2] Note that the general derivation of the πK scattering amplitude was already done in [6]

$$+ t\Bigg(16L_4^r + 4L_5^r - \frac{37}{36}\frac{1}{(4\pi)^2} - \frac{3}{4}\frac{1}{(4\pi)^2}\ln\left(\frac{\bar{M}_K^2}{\lambda^2}\right) - \frac{1}{4}\frac{1}{(4\pi)^2}\ln\left(\frac{4}{3}\frac{\bar{M}_K^2}{\lambda^2}\right)\Bigg) - \frac{3}{4}\frac{t}{F^2}J_{\pi\pi}^r(t))\Bigg]. \quad (4.2)$$

We then equate the coefficients of the various terms in (4.2) with the ones in (4.1) and arrive at the desired matching relations:

$$-4A_3^r - 8A_4^r - 1 = \frac{\bar{M}_K^2}{F^2}\Bigg(-32L_4^r - 16L_5^r + 64L_6^r + 32L_8^r + \frac{2}{9}\frac{1}{(4\pi)^2} + \frac{8}{9}\frac{1}{(4\pi)^2}\ln\left(\frac{4}{3}\frac{\bar{M}_K^2}{\lambda^2}\right)\Bigg) + \mathcal{O}(\bar{M}_K^4), \quad (4.3)$$

$$16A_3^rC_5^r + 32A_3^rC_6^r + 32A_4^rC_5^r + 64A_4^rC_6^r + \frac{8}{F^2}A_3^rl_3^r + \frac{16}{F^2}A_4^rl_3^r - 32C_{13}^r - 32C_{14}^r - 64C_{15}^r - \frac{1}{F^2}\frac{1}{(4\pi)^2}A_2^rM_K^2 = \frac{1}{F^2}\Bigg(-16L_4^r + 32L_6^r + \frac{1}{3}\frac{1}{(4\pi)^2} + \frac{5}{18}\frac{1}{(4\pi)^2}\ln\left(\frac{4}{3}\frac{\bar{M}_K^2}{\lambda^2}\right)\Bigg) + \mathcal{O}(\bar{M}_K^2), \quad (4.4)$$

$$8A_3^r + 16A_4^r + 3A_1^r + \frac{A_2^r}{2}M_K^2 = -\frac{1}{2} + \mathcal{O}(\bar{M}_K^2), \quad (4.5)$$

$$2C_5^r + 4C_6^r + \frac{1}{A_\chi^2}\frac{A_2^r}{6}M_K^2 = \frac{1}{F^2}\Bigg(16L_4^r + 4L_5^r - \frac{37}{36}\frac{1}{(4\pi)^2} - \frac{3}{4}\frac{1}{(4\pi)^2}\ln\left(\frac{\bar{M}_K^2}{\lambda^2}\right) - \frac{1}{4}\frac{1}{(4\pi)^2}\ln\left(\frac{4}{3}\frac{\bar{M}_K^2}{\lambda^2}\right)\Bigg) + \mathcal{O}(\bar{M}_K^4), \quad (4.6)$$

$$-\frac{3}{2}A_1^r - \frac{1}{4}A_2^rM_K^2 = -\frac{3}{4} + \mathcal{O}(\bar{M}_K^2), \quad (4.7)$$

$$6A_3^r + 12A_4^r + 3A_1^r + A_2^rM_K^2 = \mathcal{O}(\bar{M}_K^2). \quad (4.8)$$

These conditions, except one, have previously been obtained [6] (note that we have corrected for some obvious misprints in that paper). Equation (4.6) provides new information, whose source is essentially the t-dependence of Γ_K. To the accuracy we are working we can neglect the term proportional to the kaon mass, such that the essential information from matching the scalar form factor of the kaon is given by the following equation:

$$C_5^r + 2C_6^r = \frac{1}{F^2}\Bigg(8L_4^r + 2L_5^r - \frac{37}{72}\frac{1}{(4\pi)^2} - \frac{3}{8}\frac{1}{(4\pi)^2}\ln\left(\frac{\bar{M}_K^2}{\lambda^2}\right) - \frac{1}{8}\frac{1}{(4\pi)^2}\ln\left(\frac{4}{3}\frac{\bar{M}_K^2}{\lambda^2}\right)\Bigg) + \mathcal{O}(\bar{M}_K^2). \quad (4.9)$$

This can, in turn, be used to isolate $C_{13}^r + C_{14}^r + 2C_{15}^r$ from (4.4) with the following result:

Table 2. Values of some combinations of HKCHPT LECs for various choices of the SU(3) CHPT LECs $L_i^r(M_\rho)$. The first eight entries are derived from matching the πK scattering amplitude (some of these are also found in the analysis of the scalar kaon form factor as explained in the text). The next two stem from the momentum dependence of $\Gamma_K(t)$. The large variation for these two can be traced back to the rather different input values for some of the OZI-violating LECs in sets 1 and 2, respectively. The last two are particular combinations of dimension three LECs which can be obtained from the former relations

	set 1	set 2
A_1^r	0.68	0.52
$A_3^r + 2A_4^r$	−0.26	−0.28
A_2^r [GeV^{-2}]	−6.35	−4.68
B_1^r [GeV^{-2}]	0.93	0.56
B_3^r [GeV^{-2}]	0.83	0.68
C_1^r [GeV^{-2}]	−1.96	−0.74
$C_5^r + 2C_6^r + 4C_7^r + 2C_8^r + 4C_9^r$ [GeV^{-2}]	−2.03	−2.03
$8(C_{13}^r + C_{14}^r + 2C_{15}^r + C_{16}^r) + C_8^r + 2C_9^r$ [GeV^{-2}]	−0.84	−1.07
$C_5^r + 2C_6^r$ [GeV^{-2}]	−0.02	0.15
$C_{13}^r + C_{14}^r + 2C_{15}^r$ [GeV^{-2}]	−0.001	−0.03
$2C_7^r + C_8^r + 2C_9^r$ [GeV^{-2}]	−1.01	−1.09
$8C_{16}^r + C_8^r + 2C_9^r$ [GeV^{-2}]	−0.83	−0.82

$$C_{13}^r + C_{14}^r + 2C_{15}^r = \frac{1}{F^2}\Bigg(-2L_6^r - \frac{1}{2}L_8^r + \frac{1}{18}\frac{1}{(4\pi)^2} + \frac{3}{64}\frac{1}{(4\pi)^2}\ln\left(\frac{\bar{M}_K^2}{\lambda^2}\right) + \frac{5}{576}\frac{1}{(4\pi)^2}\ln\left(\frac{4}{3}\frac{\bar{M}_K^2}{\lambda^2}\right)\Bigg) + \mathcal{O}(\bar{M}_K^2). \quad (4.10)$$

As we did in the preceding section, we handle the theoretical uncertainties by working with two sets of values for the SU(3) CHPT LECs. The results are displayed in Table 2. They reflect the LECs of the heavy-kaon theory for a certain renormalization scale, which is inherited from the standard LECs used in the calculation, i.e. $\lambda = M_\rho$. We note that while the dimension two heavy LEC combination is well determined, there is a large variation in the dimension four heavy-kaon LEC combinations for the two sets of L_i. This is interesting because to a certain extent it reflects the dependence on the OZI-violating LECs L_4^r and L_6^r. Employing these matching conditions, the scalar form factor can be studied numerically. First, we find that the normalization $\Gamma_K(0)$ increases as compared to the standard SU(3) CHPT case, $\Gamma_K(0) = 0.56(0.61)M_\pi^2$ for set 1 (2) (see also the discussion below). Second, as a consequence of that, the corresponding radius shrinks a bit,

$$\langle r_S^2\rangle_K = 0.23(0.26)\,\text{fm}^2, \quad \text{set } 1(2); \quad (4.11)$$

compare (3.4). We refrain from repeating the analysis of the theoretical error due to the uncertainty in the heavy-kaon LECs since this would only reflect the uncertainty of the L_i^r already discussed in the preceding section. At first, the closeness of the values for the scalar radius using sets 1 or 2 seems puzzling since in the polynomial part of (4.1) the term linear in t is multiplied by $C_5^r + 2C_6^r$, which is very different for the two sets. However, these LECs are very small and furthermore, this effect is to a large portion cancelled by the contribution from the term $\sim A_1^r J^r_{\pi\pi}(t)$.

Next, we consider the πK amplitude and the remainder at the CD-point. First, we note that the heavy-kaon scattering amplitude has been first evaluated and analyzed in [6]. However, the amplitude given in that paper is not free of errors; therefore, we give the corrected form in Appendix D. With that result, the reported discrepancy [6] between the chiral prediction for some of the threshold parameters in the relativistic and the heavy-kaon scheme disappears. We have also rederived the matching relations from the amplitude, which mostly agree with the ones in [6]. In two relations, we found a discrepancy; the corrected formulae are displayed in Appendix D. The numerical results are collected in Table 2. Putting pieces together, we arrive at the remainder at the CD-point,

$$\Delta_{\pi K} = M_\pi^4 \Bigg(-\frac{A_2^r}{4} - 4(A_3^r + 2A_4^r)(C_5^r + 2C_6^r) + 4C_7^r - 16C_{16}^r \Bigg). \tag{4.12}$$

Note that all non-polynomial pieces have disappeared. This is consistent with the previous finding because for the choice $F^2 = F_\pi^2$ in the standard scenario we had no contribution from pure pion loops $\sim J^r_{\pi\pi}$ and the logarithmic terms $\ln M_\pi$ in the light-kaon case (see (3.5)) only appear at higher orders in the heavy-kaon power counting. Employing the matching relations, we can analyze the LET, (2.5), and find (again normalizing the tree result to one)

$$\begin{aligned} 1.22(1.27) &= 1.18(1.26) + 0.048(0.014), \\ F^2 A_{\pi K}^{\mathrm{CD}} &= \Gamma_K(2M_\pi^2) + \Delta_{\pi K}, \end{aligned} \tag{4.13}$$

which means that the relative size of the remainder is 3.9% (1.6%) for set 1 (2). This is similar to the results in standard SU(3) CHPT for the choice $F^2 = F_\pi^2$. We also note that the normalization of the form factor $\Gamma_K(0)$ has somewhat increased in the heavy-kaon approach. This is due to the value of $A_3^r + 2A_4^r$ which via the matching condition subsumes some higher order corrections. A similar statement can be made for the pion–kaon amplitude. At first sight, this might appear worrisome but it can be traced back to our treatment of the matching conditions, on which we imposed a strict power counting in $\bar{M}_K$. It would of course also be allowed to include such higher order terms in the matching conditions. This would lead to a reduction of the apparent discrepancy between the heavy-kaon and the standard formulation. However, our intention in using the heavy-kaon formulation was not to reproduce exactly the numbers obtained in the standard case but rather to consider the same observables in a scheme

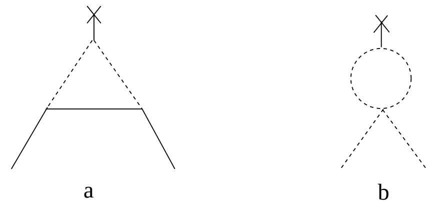

Fig. 1a,b. Dominant loop contributions to the scalar form factor. **a** In the pion–nucleon case, this stems from the so-called triangle graph. Solid (*dashed*) lines denote nucleons (pions). **b** In the pion–kaon case, one has a tadpole-like contribution and others. Here, the dashed lines denote Goldstone bosons

which treats the kaons very differently. Furthermore, these small isoscalar observables are also subject to the largest theoretical uncertainties, a situation similar to the case of pion–nucleon scattering. However, it is also important to discuss the difference to the pion–nucleon scattering amplitude. So far, we have stressed the similiarity between πK and πN scattering, but there are some differences due to the absence of three-Goldstone-boson couplings. In the context discussed here, this has a major influence on the momentum dependence of the scalar form factor respectively on the t-dependence of the scattering amplitude. In the pion–nucleon case, the very strong momentum dependence around $t = 4M_\pi^2$ is due to the fact that the so-called triangle diagram (see Fig. 1a) has a singularity on the second Riemann sheet at $t_c = 4M_\pi^2 - M_\pi^4/m^2 = 3.98M_\pi^2$, i.e. very close to the threshold. In fact, in the heavy fermion limit, this singularity coalesces with the threshold and thus distorts the analytical structure. Such an effect can also be seen in the spectral functions of the isovector nucleon form factors. Quite differently, the t-dependence for the pion–kaon case is given by loop graphs (as shown in Fig. 1b) and contact terms that do not have such close-by singularities. We end this section by remarking that one may also try to fix the heavy-kaon LECs A_i^r, B_i^r and C_i^r directly from a systematic analysis of low-energy data involving kaons. Given however the scarcity of precise data for processes with a conserved kaon number, we refrain from performing such an analysis here.

5 Two-loop representation of Γ_K

In Sect. 3 we have discussed the one-loop representation of the scalar form factor Γ_K of the kaon in the framework of SU(3) CHPT. Numerically, the fourth order contributions were found to be about 10% at the CD-point $t = 2M_\pi^2$ and 22% at the two-pion threshold $s = 4M_\pi^2$. We have also shown that the small correction to the LET (2.5) was due to the suppression of terms proportional to powers of the kaon mass when the field normalization $\sim 1/F_\pi^2$ was chosen. Still, in view of possible large higher order corrections in the S-wave isospin zero channel and the expected slow convergence behavior of three flavor chiral perturbation theory, it is mandatory to estimate the two-loop contributions. Because two-loop diagrams are awkward to calculate, we seek to obtain information concerning contributions of sixth chiral order at lower cost by following a different strategy, which is based on the unitarity properties of the scattering operator and on the analyticity properties of the perturbation series representing Γ_K. We follow essentially the work of [9] on the dispersive representation of the pion form factors. We do not perform a very precise determination of the occurring subtraction constants. For our purpose, however, this procedure is of sufficient accuracy.

The form factor Γ_K can be represented by means of an n-fold subtracted dispersion relation, which, restricted to the real axis below respectively above the upper rim of the two-pion cut beyond threshold, reads

$$\Gamma_K(s+\mathrm{i}\epsilon) = \sum_{i=0}^{n-1} a_i s^i + \frac{s^n}{\pi}\int_{4M_\pi^2}^{\infty} \frac{\mathrm{d}s'}{s'^n} \frac{\mathrm{Im}(\Gamma_K(s'+\mathrm{i}\epsilon))}{s'-s-\mathrm{i}\epsilon}, \quad \epsilon \to 0^+, \tag{5.1}$$

where the a_i are subtraction constants, whose number n is dictated by the convergence behavior of Γ_K at infinity, and the $s+\mathrm{i}\epsilon$ notation indicates that, for $s > 4M_\pi^2$, we evaluate Γ_K at the upper edge of the branch cut. From quark counting rules, one expects the real (imaginary) part of Γ_K to fall off as $1/s$ ($1/s^2$). The central object in the dispersion relation (5.1) is the absorptive part, which can expressed via

$$\mathrm{Im}\Gamma_K(s) = \frac{\mathrm{i}}{2}\sum_n \langle K^+(p_3), K^+(p_1)\mathrm{in}|\mathcal{T}^\dagger|\mathrm{in}\, n\rangle\langle n\,\mathrm{in}|\mathcal{T}J|0\rangle, \tag{5.2}$$

where the summation extends over the complete set of intermediate states $|n\rangle\langle n|$, i.e. including all sorts of multiparticle states with appropriate quantum numbers to satisfy the pertinent conservation laws[3]. Furthermore, $J = \hat{m}(\bar{u}u + \bar{d}d)$ is the scalar–isoscalar source (current) under consideration, and we have made use of the $\mathcal{T}$-operator, which is the non-trivial part of the $\mathcal{S}$-operator transforming a state $|\mathrm{in}\rangle$ from the Fock space of incoming states into an outgoing state $|\mathrm{out}\rangle$, $\mathcal{S}|\mathrm{in}\rangle = (1+\mathrm{i}\mathcal{T})|\mathrm{in}\rangle = |\mathrm{out}\rangle$. The second term on the right-hand side of (5.2) is nothing but the complex conjugate of that scalar form factor describing the coupling of the source J to the particles of the intermediate state labeled n, while the first term is essentially the amplitude associated with two-kaon scattering into this particular intermediate state. We have thus reexpressed the imaginary part of Γ_K in terms of various form factors and scattering amplitudes. In an order by order analysis it follows that the lowest order imaginary part of the scalar form factor is of order $\mathcal{O}(p^4)$, since on the right-hand side of (5.2) there are two quantities of at least second chiral order. More generally speaking, $\mathrm{Im}\Gamma_K$ to any order d in the energy expansion is related to $\mathrm{Re}\Gamma_K$ to order $d-2$. $\mathrm{Im}\Gamma_K$ to order d is completely determined by the Lagrangian terms up to order $d-2$ via (5.2). Therefore, once its imaginary part to order d is known, we can, within the analyticity domain, recover the order $\mathcal{O}(p^d)$ contribution of Γ_K up to a number of subtraction constants by invoking the analyticity properties of its perturbative expansion. To leading order in the chiral expansion, we have to consider two-particle intermediate states in (5.2). It is well established from phenomenology that four particle (pion) intermediate states only play a role for energies above about 1.3 GeV and will thus be neglected in what follows. For the case under consideration, the following isospin zero states made from two equal Goldstone bosons must be considered,

$$\begin{aligned} \pi^+\pi^- + \pi^-\pi^+ - \pi^0\pi^0 &= -\sqrt{3}|0,0\rangle, \\ K^+K^- - K^-K^+ - \overline{K^0}K^0 + K^0\overline{K^0} &= 2\sqrt{2}|0,0\rangle, \\ \eta\eta &= |0,0\rangle. \end{aligned} \tag{5.3}$$

Performing furthermore the S-wave projection of the corresponding $KK \to \pi\pi, KK, \eta\eta$ scattering amplitudes (since we are dealing with a scalar source), the imaginary part of Γ_K at sixth chiral order can finally be written as

$$\begin{aligned} \mathrm{Im}\Gamma_K^{(6)}(s) = &-\sqrt{\frac{3}{2}}\Big(t^{0,(2)}_{0,KK\to\pi\pi}\mathrm{Re}\Gamma_\pi^{(4)} \\ &+ \mathrm{Re}\left(t^{0,(4)}_{0,KK\to\pi\pi}\right)\Gamma_\pi^{(2)}\Big)\Sigma_\pi(s)\Theta(s-4M_\pi^2) \\ &+ 2\Big(t^{0,(2)}_{0,KK\to KK}\mathrm{Re}\Gamma_K^{(4)} \\ &+ \mathrm{Re}\left(t^{0,(4)}_{0,KK\to KK}\right)\Gamma_K^{(2)}\Big)\Sigma_K(s)\Theta(s-4M_K^2) \\ &+ \frac{1}{\sqrt{2}}\Big(t^{0,(2)}_{0,KK\to\eta\eta}\mathrm{Re}\Gamma_\eta^{(4)} \\ &+ \mathrm{Re}\left(t^{0,(4)}_{0,KK\to\eta\eta}\right)\Gamma_\eta^{(2)}\Big)\Sigma_\eta(s)\Theta(s-4M_\eta^2), \end{aligned} \tag{5.4}$$

where we have generalized our definitions of the (nonstrange) scalar form factors according to

$$\langle \Phi_a(p_3), \Phi_b(p_1)\mathrm{out}|J|0\rangle = \delta_{ab}\Gamma_a(s). \tag{5.5}$$

For $a \in \{4,5,6,7\}$, (5.5) coincides with the earlier definition of Γ_K. The scalar form factor Γ_π of the pion corresponds to $a \in \{1,2,3\}$, where there is no need to distinguish these three cases in the isospin symmetric case. The scalar form factor Γ_η of the eta results from the choice $a = 8$, when mixing of Φ_3 and Φ_8 is neglected. The superscripts $(n)(n = 2,4)$ refer to the chiral order, that is to tree level (2) and one-loop accuracy (4). Furthermore, $\Sigma_a(s) = (1-4M_a^2/s)^{1/2}$ and t_0^0 denotes the corresponding $l = 0$, $I = 0$ scattering amplitudes. A graphic illustration of this formula in provided by Fig. 2.

From this formula, the ingredients necessary for the calculation of $\mathrm{Im}\Gamma_K$ can be read off: Besides Γ_K, we need the scalar form factors of the pion and of the eta to one-loop order, furthermore the S-wave projections of the isospin zero amplitudes $T^0_{KK\to\pi\pi}$, $T^0_{KK\to KK}$, and $T^0_{KK\to\eta\eta}$ to $\mathcal{O}(p^4)$. All these quantities are listed in Appendix E. Note that for calculating the imaginary part of Γ_K given in (5.4) we need the $KK \to \pi\pi$ scattering amplitude in the unphysical region $s \in [4M_\pi^2, 4M_K^2]$. The amplitude can be

[3] Note that from here on we label the incoming momenta as p_1 and p_3 and the out-going ones as p_2 and p_4, see also Appendix E

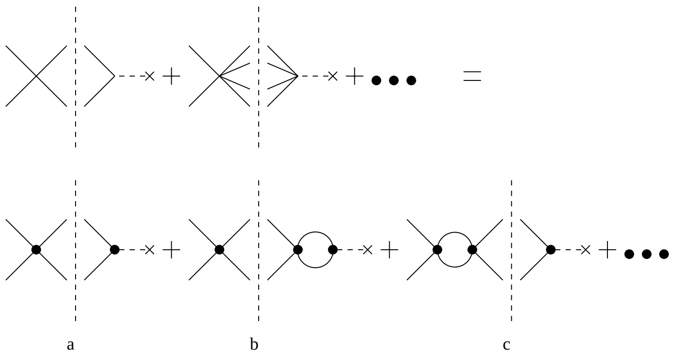

Fig. 2a–c. Imaginary part of Γ_K. The $\times$ denotes the coupling to the scalar–isoscalar source

reconstructed in this regime by means of an Omnès representation, as treated in detail in [17]. We refer to that paper for all details and simply apply the same procedure.

In what follows, we chose to work with a triple subtracted dispersion relation for the scalar kaon form factor $\Gamma_K(s)$. Therefore, the normalization, the radius and the curvature terms appear in the polynomial part of the dispersive representation, which allows for the most transparent way of fixing the various subtraction constants (LECs). We thus have

$$\Gamma_K^{(4+6)}(s+\mathrm{i}\epsilon) = P(s) + \frac{s^3}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}s'}{s'^3}\frac{\mathrm{Im}\Gamma_K^{(4+6)}(s'+\mathrm{i}\epsilon)}{s'-s-\mathrm{i}\epsilon}, \tag{5.6}$$

with the polynomial

$$\begin{aligned} P(s) &= P_4(s) + P_6(s) \\ &= \frac{M_\pi^2}{\Lambda_\chi^2}\left(\left(d_1 M_\pi^2 + \frac{d_2 M_\pi^4}{\Lambda_\chi^2}\right) + \left(f_1 + \frac{f_2 M_\pi^2}{\Lambda_\chi^2}\right)s + \frac{g}{\Lambda_\chi^2}\frac{s^2}{2}\right). \end{aligned} \tag{5.7}$$

Here, the dimensionless numbers $d_1, f_1 (d_2, f_2, g)$ are combinations of dimension four (six) LECs. In what follows, we will fix d_1 and f_1 from the normalization and radius at one-loop accuracy and set $d_2 = f_2 = 0$ for our central results. We will also vary the latter two within reasonable bounds, $\Delta d_2 = \Delta f_2 = \pm 1/(16\pi)^2$. The coupling g can be determined from the requirement that the normalized scalar form factor Γ_K/M_π^2 stays finite in the chiral limit (cl). Setting $m_u = m_d = m_s = 0$, we find the following representation of the sixth order contribution to this quantity:

$$\begin{aligned} &\frac{\Gamma_K^{(6),\mathrm{cl}}(s)}{M_\pi^2} \\ &= \frac{1}{(4\pi F)^4}s^2\left((4\pi)^2\left(\frac{2632}{45}L_1^r + \frac{3082}{135}L_2^r + \frac{8773}{405}L_3^r\right.\right. \\ &+ \left.\frac{70}{3}L_4^r + \frac{1012}{135}L_5^r - \frac{428}{45}L_6^r - \frac{85}{27}L_8^r\right) \\ &+ (4\pi)^2\left(\frac{68}{3}L_1^r + \frac{88}{9}L_2^r + \frac{232}{27}L_3^r + 12L_4^r + \frac{7}{2}L_5^r\right) \\ &\times \left(\ln\left(\frac{M^2}{\lambda^2}\right) + \ln\left(\frac{\lambda^2}{-s}\right)\right) \\ &+ \frac{g}{2} + \gamma + \frac{17761}{12960} - \frac{97871}{25920}\ln\left(\frac{M^2}{\lambda^2}\right) - \frac{325}{192}\ln^2\left(\frac{M^2}{\lambda^2}\right) \\ &+ \left.\frac{661}{192}\ln\left(\frac{\lambda^2}{-s}\right) + \frac{325}{192}\ln^2\left(\frac{\lambda^2}{-s}\right)\right), \end{aligned} \tag{5.8}$$

where the constant $2\gamma/(4\pi F)^4$ is the part of the second derivative of the absorptive part of the dispersive representation of Γ_K with respect to s generated by the terms in the S-wave projected scattering amplitudes which we could only represent in an integral form; see Appendix E. Also, the arguments of the logarithms $\sim M^2$ have been made dimensionless by the square of the renormalization scale λ. The requirement that Γ_K stays finite in the chiral limit implies that the chiral logarithms are compensated by corresponding terms in g:

$$\begin{aligned} g &= g^r - (4\pi)^2\left(\frac{136}{3}L_1^r + \frac{176}{9}L_2^r\right. \\ &+ \left.\frac{464}{27}L_3^r + 24L_4^r + 7L_5^r\right)\ln\left(\frac{M^2}{\lambda^2}\right) + \frac{97871}{12960}\ln\left(\frac{M^2}{\lambda^2}\right) \\ &+ \frac{325}{96}\ln^2\left(\frac{M^2}{\lambda^2}\right), \end{aligned} \tag{5.9}$$

where parts of γ have been reshuffled to the finite constant g^r. g is thus found to contain chiral logarithms and squared chiral logarithms together with a finite part. This structure reflects the singularity structure of g before renormalization: In dimensional regularization, g absorbs poles of first and of second order in $1/(d-4)$, with the related chiral logarithms restoring independence on the renormalization scale. The scale dependence of g is given by

$$\begin{aligned} \frac{\partial g}{\partial\lambda} &= \frac{\partial g^r}{\partial\lambda} + \frac{1}{\lambda}\left((4\pi)^2\left(\frac{272}{3}L_1^r + \frac{352}{9}L_2^r + \frac{928}{27}L_3^r\right.\right. \\ &+ \left.\left.48L_4^r + 14L_5^r\right) - \frac{97871}{6480}\right), \end{aligned} \tag{5.10}$$

i.e. the derivative of the double chiral logarithm is canceled by the contributions of the L_i. Similarly, the logarithmic scale dependence of the L_i balances the scale dependence of g^r, $\lambda\partial g^r/\partial\lambda$, when requiring $\partial g/\partial\lambda = 0$. Equation (5.9) allows one to estimate the coupling g. Neglecting g^r, we evaluate g for $\lambda = M_\rho$. Identifying the meson mass M with the pion (eta) mass, we find $g = 6.7(-5)$. This ambiguity is to be contrasted with the two flavor case, where only the pion mass can appear, and thus the corresponding constant can be fixed unambiguously [9]. Since the chiral pion loops are longer ranged than kaon or eta loops (and are thus more important), it is however reasonable to set $M = M_\pi$ also in the SU(3) case as will be done in what follows.

The real and imaginary parts of the normalized (non-strange) scalar kaon form factor Γ_K are shown in Fig. 3. Consider first the real part. The overall correction to the tree level result at the two-pion threshold $s = 4M_\pi^2$ amounts to 23%, with only 5% due to two-loop effects. For larger energies, the two-loop result turns over while

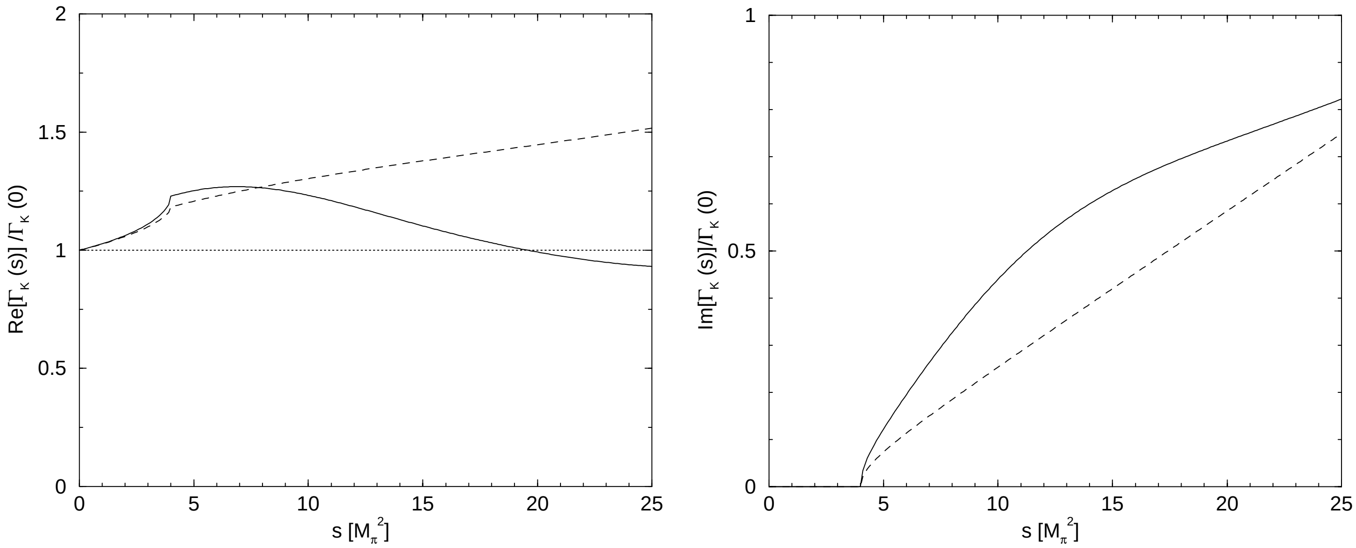

Fig. 3. Normalized scalar form factor $\Gamma_K(s)/\Gamma_K(0)$. Left (right) panel: real (imaginary) part. The dotted, dashed and solid lines represent the tree, one-loop and two-loop result, in order

the one-loop curve keeps on rising. This is similar to the case of the pion scalar form factor [9]. The turn-over of the two-loop curve is due to the one-loop phase passing through 90° in the region of the scalar resonances. If one varies the constants d_2 and f_2 as described before, the real part is only mildly affected. Its decrease for $s \geq 10M_\pi^2$ is much steeper if we chose to set $M = M_\eta$ in the determination of the constant g. However, at $s = 4M_\pi^2$, this different choice of g only reduces the two-loop correction to about 3%. The imaginary part is only non-zero at one-loop order and the corrections from the two-loop graphs are more sizable in the threshold region as shown in the right panel of Fig. 3. This is very similar to the case of the scalar pion form factor studied in [9]. Note also that the final state interactions are weaker in the pion–kaon system than in the pion case (as signalled e.g. by the mass of the dynamically generated light scalar mesons in the two channels, see e.g. [18]). We stress again the difference to the case of pion–nucleon scattering. There, the momentum dependence is much stronger (for the reasons discussed in the previous section), despite the apparent similarity to the case of pion–kaon scattering considered in this work.

6 Summary

To summarize, we have considered aspects related to scalar form factors and pion–kaon scattering in chiral perturbation theory. More precisely, the pertinent results of this investigation can be summarized as follows.

(1) We have analyzed the low-energy theorem (2.5) for pion–kaon scattering. The remainder at the Cheng-Dashen point turns out to be much smaller than expected from naive dimensional analysis in three flavor chiral perturbation theory. In particular, setting the meson decay constant $F = F_\pi$, the remainder is comparable to the one in pion–pion scattering [3].

(2) We have shown that the result for the remainder can be understood in terms of approximate scale relations by representing the one-loop corrections to the scalar kaon form factor and the πK scattering amplitude in terms of chiral logarithms with appropriate scales Λ_i and $\tilde{\Lambda}_i$, $i = 1, 2$. These scales are found to be close to the eta mass, thus suppressing the potentially large chiral logarithms multiplying the kaon mass squared.

(3) We have repeated the analysis in the heavy-kaon framework, in which the kaons are treated as matter fields. Matching conditions allow one to fix the new low-energy constants from the ones based on the standard chiral expansion with light kaons. We have performed this matching procedure for the scalar kaon form factor and the πK scattering amplitude. This analysis confirms the finding in the standard approach. Since a heavy-kaon cannot decay, no ambiguity arises as to the choice of the meson decay constant.

(4) Since the pion scalar form factor is subject to large two-loop corrections already close to threshold, we have calculated these also for the scalar kaon form factor using a dispersive representation [9]. The pertinent subtraction constants are fixed from the one-loop representation of this form factor and the condition that it is well defined in the chiral limit. The resulting two-loop corrections for the real part are fairly small at low energies, while they are more pronounced for the imaginary part. We have also discussed the dependence of the scalar kaon radius on the OZI-violating low-energy constant L_4.

Acknowledgements. We are grateful to Jürg Gasser for some valuable comments and encouragement. We also thank Paul Büttiker for tuition on subthreshold scattering amplitudes.

Appendix

A Low-energy kaon relations

In very much the same manner in which we have analyzed the isospin-even πK amplitude in the regime of low pion momenta, we can also consider the limit of vanishing kaon momenta. To be specific, we consider again the reaction $K^+(p_1)+\pi^+(p_3) \to K^+(p_2)+\pi^+(p_4)$, but now in the soft-kaon limit of vanishing kaon four-momenta. The resulting low-energy theorem for the on-shell amplitude reads

$$T^{+}_{\pi K\to\pi K}(\nu=0,t=2M_K^2) \equiv \tilde{A}^{\mathrm{CD}}_{\pi K} = \frac{1}{F^2}\tilde{\varGamma}_{\pi^+}(2M_K^2)+\mathcal{O}(M_K^4), \qquad \text{(A.1)}$$

where the strange form factor of the pion, $\tilde{\varGamma}_\pi$, is defined as follows:

$$\tilde{\varGamma}_\pi(t) = \langle\pi^+(p_4)|\frac{1}{2}(\hat{m}+m_s)(\bar{u}u+\bar{s}s)|\pi^+(p_3)\rangle, \qquad \text{(A.2)}$$

with $t=(p_3-p_4)^2$. The analogue of the CD-point is now the kinematic configuration $\nu=0, t=2M_K^2$ far off the physical region of elastic scattering. That makes the experimental determination of $\tilde{\varGamma}_\pi$ more involved than in the case of $\varGamma_K$ since data have to be extrapolated further beyond the physical domain. The one-loop calculation of $\tilde{\varGamma}_\pi$ is straightforward and leads to

$$\begin{aligned}\tilde{\varGamma}_{\pi^+}(t) = \frac{M_K^2}{2} + \frac{M_K^2}{2F^2}\Bigg[& L_4^r(-32M_\pi^2+16t) \\ & + L_5^r(-16M_\pi^2+8M_K^2+4t)+L_6^r 64M_\pi^2 \\ & + L_8^r(32M_\pi^2-16M_K^2) \\ & + \frac{1}{2}M_\pi^2\mu_\pi + \left(\frac{1}{6}M_\pi^2-\frac{2}{3}M_K^2\right)\mu_\eta + \left(\frac{1}{2}M_\pi^2-t\right)J^r_{\pi\pi}(t) \\ & - \frac{3}{4}tJ^r_{KK}(t) - \frac{5}{18}M_\pi^2 J^r_{\eta\eta}(t)\Bigg] + \mathcal{O}(p^6). \end{aligned} \qquad \text{(A.3)}$$

Note that the OZI-violating coupling L_4 contributes significantly to the form factor for four-momenta of the order $t \simeq M_K^2$, but is suppressed in the low-energy region much as the other OZI-violating LEC, L_6, as it only appears with a prefactor $\sim M_\pi^2$. The first moment of the low momentum expansion of this form factor is given in terms of the pertinent strange pion radius,

$$\langle \tilde{r}_{\mathrm{S}}^2\rangle_\pi = \begin{cases}(0.41\pm0.22)\,\mathrm{fm}^2, & \text{set } 1,\\ (0.49\pm0.02)\,\mathrm{fm}^2, & \text{set } 2.\end{cases} \qquad \text{(A.4)}$$

whose central value is smaller than the corresponding non-strange radius of about $0.6\,\mathrm{fm}^2$. This pattern is to be expected since the strange quark is more massive than the light quarks and thus leads to smaller scales in coordinate space. Concerning the theoretical uncertainty, the same remarks as after (3.4) apply here. In analogy with our previous considerations, we examine the validity of the low-energy theorem (2.5) in terms of the remainder $\tilde{\varDelta}_{\pi K}$:

$$F^2\tilde{A}^{\mathrm{CD}}_{\pi K} = \tilde{\varGamma}_\pi(2M_K^2)+\tilde{\varDelta}_{\pi K}. \qquad \text{(A.5)}$$

Table 3. Low-momentum kaon theorem. Size of the remainder $\tilde{\varDelta}_{\pi K}$ for various choices of the meson decay constants and the low-energy constants $L_i^r(M_\rho)$

F^2	L_i set	$F^2\tilde{A}^{\mathrm{CD}}_{\pi K}$ $[M_K^2]$	$\tilde{\varGamma}_\pi(2M_K^2)$ $[M_K^2]$	$\tilde{\varDelta}_{\pi K}[M_K^2]$	R [%]
F_π^2	1	$0.779+0.510\mathrm{i}$	$0.825+0.510\mathrm{i}$	-0.046	4.9
$F_\pi F_K$	1	$0.886+0.510\mathrm{i}$	$0.825+0.510\mathrm{i}$	0.061	6.0
F_K^2	1	$0.993+0.510\mathrm{i}$	$0.825+0.510\mathrm{i}$	0.168	15.1
F_π^2	2	$0.813+0.510\mathrm{i}$	$0.917+0.510\mathrm{i}$	-0.104	10.8
$F_\pi F_K$	2	$0.894+0.510\mathrm{i}$	$0.917+0.510\mathrm{i}$	-0.021	2.2
F_K^2	2	$0.975+0.510\mathrm{i}$	$0.917+0.510\mathrm{i}$	0.058	5.3

Since we are now working above the two-pion threshold $t=4M_\pi^2$, we will generally have to deal with imaginary contributions. As it turns out, these cancel exactly in the low-momentum kaon relation, such that $\tilde{\varDelta}_{\pi K}$ is real. The numerical results for the amplitude and for $\tilde{\varGamma}_\pi$ at $\nu=0, t=2M_K^2$ are displayed in units of M_K^2 in Table 3, together with the remainder in units of M_K^2 and its relative size R, defined as the ratio of $|\tilde{\varDelta}_{\pi K}|$ and the complex modulus of the amplitude at $\nu=0, t=2M_K^2$. R is given in %.

The situation is somewhat different compared to the previous case. For the LECs from set 1, the best agreement is given for the normalization $1/F_\pi^2$, where the relative deviation amounts to about 5%. The situation is different when working with the LECs from set 2, where the decay constant combination $1/F_\pi F_K$ accounts for the smallest remainder of about 2% relative size. Both of these values for R are larger than the corresponding values in the low-energy pion case, as is expected due to the larger kaon mass. Also, the dependence on the choice of decay constant is less pronounced. We note, however, that the deviations from the LET are fairly small for all choices of F^2 and are well below the typical SU(3) corrections $M_K^2/\varLambda_\chi^2 \simeq 0.2$.

B Basics of heavy-kaon chiral perturbation theory

The basic concepts of heavy-kaon CHPT (HKCHPT) are adopted from heavy baryon CHPT, as introduced in [19] to include baryon fields into the framework of chiral perturbation theory. Since baryon masses are comparable to the chiral symmetry breaking scale and are non-vanishing in the chiral limit, they cannot be considered light. As a consequence hard momenta enter into the theory and the standard power counting scheme breaks down. This is because arbitrarily complicated diagrams no longer yield contributions of a fixed chiral order, but contributions of any lower order are now possible if only a sufficient number of momenta is provided by derivatives acting on the heavy fields. Heavy baryon CHPT therefore treats baryons essentially as static in an extreme non-relativistic framework with small residual (that is, soft) momenta. In the standard approach, see e.g. [20], a baryon field $B(x)$ is rewritten in the form

$$B(x) = \mathrm{e}^{-\mathrm{i}mv\cdot x} b(x), \tag{B.1}$$

where m is the baryon mass and v_μ a four-velocity obeying $v^2 = 1$. The field b has only small residual momentum which can be treated on an equal footing with the other generically small momenta and masses, p. Using (B.1) for the heavy fields, one can perform an expansion in powers of $1/m$. The result is a Lagrangian which generally breaks Lorentz invariance and gives rise to a modified propagator with additional new vertices suppressed by powers of $1/m$. These can be included in a power counting scheme where contributions are organized in terms of both powers of p/Λ_χ and of p/m. Since Λ_χ and m are of the same order of magnitude, it is not necessary to differentiate between these various types of contributions.

In a similar way, consider now the kaon to be a heavy particle on the pionic mass scale, and apply a similar scheme to pion–kaon reactions. As before, the heavy mass scale (now the kaon mass) has to be eliminated to allow for a consistent power counting (if one uses conventional dimensional regularization). We will closely follow the approach presented in [6], correcting for a number of apparent misprints, and adding some new results. It is clear that a theory which treats pions as light, relativistic particles and kaons as heavy, non-relativistic ones cannot respect SU(3) symmetry. The pertinent symmetry group will therefore be $\mathrm{SU}(2)_V\times\mathrm{SU}(2)_A$. We therefore have to choose different representations for the pion and the kaon fields and construct the most general Lagrangian compatible with the symmetries of QCD, where again chiral symmetry plays a prominent role. Lorentz invariance will require special attention. Since the kaon now plays the role of any matter field in a theory with non-linearly realized chiral symmetry, it is natural to apply the CCWZ formalism [21]. For doing that, we combine the kaon fields into a representation as isospin doublets :

$$K = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \quad \tilde{K} = \begin{pmatrix} -\overline{K^0} \\ K^- \end{pmatrix}. \tag{B.2}$$

In what follows, we will write K as a generic symbol for any of these doublets (and call them kaon fields). The advantage of this representation is that the compensator field h provides a natural way to define the action of $\mathrm{SU}(2)_R\times\mathrm{SU}(2)_L$ on the kaon fields:

$$K(x) \to h(R, L, u(x))K(x) \tag{B.3}$$

for $R/L \in\mathrm{SU}(2)_{R/L}$ and $u^2(x) = U(x)$ parameterizes the Goldstone boson fields. Note that for pure vector transformations $R = L$, h simplifies to $h = R = L$, so that $\mathrm{SU}(2)_V$ is represented fundamentally on K. A striking difference in this treatment of the Goldstone bosons is the non-linear representation for the pion as against the linear one for the kaon degrees of freedom. While the former allows for dealing with multi-pion couplings by an expansion in powers of the relevant fields, from the latter it immediately follows that the theory will fall into separate sectors marked by the occurrence of a fixed number of kaons, therefore the effective Lagrangian $\mathcal{L}_{\mathrm{HKCHPT}}$ can be written as a string of terms:

$$\mathcal{L}_{\mathrm{HKCHPT}} = \mathcal{L}_\pi + \mathcal{L}_{\pi KK} + \mathcal{L}_{\pi KKKK} + \ldots \tag{B.4}$$

While the first term describes purely pionic processes, the second one is bilinear in the kaon field, and the third one is quadrilinear, and so on. In this paper, we only consider processes with one in-coming and one out-going K, i.e. only the first two terms in this series will be of relevance. In this framework closed kaon loops, i.e. loops formed by internal K lines only, are prohibited, and their effects enter implicitly, being absorbed into the coupling constants. Kaon propagators do, however, show up in loops composed of both π and K internal lines and thus the large mass scale M_K destroys the power counting. To remedy this, one could proceed as outlined above, i.e. go over to the extreme non-relativistic limit via a field transformation analogous to (B.1),

$$K(x) = \mathrm{e}^{-\mathrm{i}M_K v\cdot x} k(x), \tag{B.5}$$

and give up Lorentz invariance right on the Lagrangian level. Diagrams are then calculated in a non-relativistic framework, and Lorentz invariance is invoked at a later stage to determine a number of relations among the coupling constants. Indeed Roessl [6] lists the most general Lagrangian in the fields u and k up to fourth order in small momenta, compatible with the symmetries of QCD except for Lorentz invariance. However, to perform calculations a different approach is proposed (such a modified scheme has also been applied in calculations of heavy baryon CHPT): one determines the manifest Lorentz invariant Lagrangian in terms of the fields u and K which generates those non-relativistic ones via the relation (B.5). Calculations are then relativistically invariant at any stage up to the evaluation of loop integrals. Only then the heavy particle expansion in $1/M_K$ is performed in those integrands containing heavy propagators, i.e. integrals of the type (as a typical example, consider a loop function with one pion and one (heavy-) kaon propagator):

$$J_{\pi K}((p_1 - p_2)^2) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{i}}{k^2 - M_\pi^2} \times \frac{1}{(p_1 - p_2 - k)^2 - M_K^2}. \tag{B.6}$$

Since this expression is Lorentz invariant, one is free to work in a frame where the incoming kaon momentum, say p_2, is of the form $p_2 = M_K v = (M_K, 0, 0, 0)$. Plugging this into the integral and expanding the integrand in powers of $1/M_K$ then yields

$$J_{\pi K}((p_1 - p_2)^2) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{i}}{k^2 - M_\pi^2} \times \left(-\frac{1}{2v\cdot(p_1 - k)} \frac{1}{M_K} - \frac{(p_1 - k)^2}{4[v\cdot(p_1 - k)]^2} \frac{1}{M_K^2} + \ldots \right), \tag{B.7}$$

where the ellipsis indicates higher powers in $1/M_K$. So one ends up with a series of terms organized as an expansion in p/M_K, where p is a generic small CHPT scale. On a diagrammatic level these can be represented by absorbing the first contribution into a modified propagator, and treating the remaining ones as additional vertices of proper order. We can then arrange any perturbative expansion derived from $\mathcal{L}_{\rm HKCHPT}$ as a dual expansion in powers of both p/Λ_χ and p/M_K. The pertinent power counting rules can be easily derived. Consider the amplitude $\mathcal{A}$ of an arbitrary graph consisting of $V_n^{\pi\pi}$ pionic vertices of order n, $V_m^{\pi K}$ pion–kaon vertices of order m, E^π external pion legs, E^K external kaon lines, I^π internal pion lines, I^K internal kaon lines, and L loops. The chiral dimension ν assigned to such a diagram is (that is, $\mathcal{A}\sim p^\nu$)

$$\nu=\sum_n V_n^{\pi\pi}(n-2)+\sum_m V_m^{\pi K}(m-1)+2L+1, \quad \text{(B.8)}$$

where we have used the topological identities $I^K=\sum_m V_m^{\pi K}-1$ and $I^\pi+I^K=L+\sum_n V_n^{\pi\pi}+\sum_m V_m^{\pi K}-1$. From this equation one readily deduces that in contrast to standard SU(3) CHPT, diagrams with odd chiral dimensions are allowed in HKCHPT. The advantage of this scheme over SU(3) CHPT lies in its improved convergence properties. For energies of the order of the pion mass, the HKCHPT expansion parameter is given by $M_\pi/M_K\approx 0.28$, thus a diagram of order $d+2$ is suppressed relative to an order d contribution by at least a factor $M_\pi^2/M_K^2\approx 0.08$. This is substantially more favorable than the corresponding minimal suppressing factor $M_K^2/\Lambda_\chi^2\approx 0.2$ in SU(3) CHPT. However desirable this feature may be, it is achieved at the price of a larger number of unknown LECs in the Lagrangian.

The difference between the heavy-kaon and the standard chiral expansion can be clearly seen in case of the pion and the kaon masses. The pion mass takes the canonical form,

$$M_\pi^2=M_0^2\left(1+2\frac{M_0^2}{F^2}l_3^r+\frac{M_0^2}{32\pi^2F^2}\ln\frac{M_0^2}{\lambda^2}\right), \quad \text{(B.9)}$$

with $M_0^2=2\hat{m}B_0$ the leading term in the quark mass expansion of the pion mass. Of course, at next-to-leading order, the SU(2) LEC l_3^r appears [22]. The kaon mass appears quadratically in the heavy-kaon Lagrangian and takes the form

$$M_K^2=M^2+M^{(2)}M_0^2+M^{(4)}M_0^4, \quad \text{(B.10)}$$

where M^2 is the quark mass independent contribution and the explicit form of the coefficients $M^{(2,4)}$ is given in [6]. Note that the first two terms in (B.10) are not renormalized and thus are finite. For the matching with the relativistic formulation one has to expand M^2, $M^{(2)}$ and $M^{(4)}$ in powers of $\bar{M}_K^2=m_sB_0$. Similarly, since in the heavy-kaon theory one only has pion loops, loop functions like e.g. J_{KK}^r and $J_{\eta\eta}^r$ must be expanded in inverse powers of $\bar{M}_K^2$,

$$J_{KK}^r(t)=\frac{1}{(4\pi)^2}\left(1+\ln\left(\frac{\bar{M}_K^2}{\lambda^2}\right)+\frac{M_\pi^2}{2\bar{M}_K^2}-\frac{t}{6\bar{M}_K^2}\right)+\mathcal{O}\left(\frac{p^4}{\bar{M}_K^4}\right),$$

$$J_{\eta\eta}^r(t)=\frac{1}{(4\pi)^2}\left(1+\ln\left(\frac{4}{3}\frac{\bar{M}_K^2}{\lambda^2}\right)+\frac{M_\pi^2}{4\bar{M}_K^2}-\frac{t}{8\bar{M}_K^2}\right)+\mathcal{O}\left(\frac{p^4}{\bar{M}_K^4}\right). \quad \text{(B.11)}$$

For more details on the heavy-kaon approach, we refer to [6,7].

C Heavy-kaon CHPT Lagrangian

First, we give the basic building blocks of the heavy-kaon Lagrangian and the associated transformation properties under chiral $R/L\in SU(2)_{R/L}$:

$$\begin{aligned}
U&=u^2, & U&\to RUL^\dagger,\\
D_\mu K&=\partial_\mu K+\Gamma_\mu K, & D_\mu K&\to hD_\mu K,\\
D_{\mu\nu}U&=(D_\mu D_\nu+D_\nu D_\mu)U, & D_{\mu\nu}U&\to RD_{\mu\nu}UL^\dagger,\\
D_{\mu\nu}K&=(D_\mu D_\nu+D_\nu D_\mu)K, & D_{\mu\nu}K&\to hD_{\mu\nu}K,\\
\Delta_\mu&=\frac{1}{2}u^\dagger D_\mu U u^\dagger, & \Delta_\mu&\to h\Delta_\mu h^\dagger,\\
\Delta_{\mu\nu}&=\frac{1}{2}(D_\mu\Delta_\nu+D_\nu\Delta_\mu), & \Delta_{\mu\nu}&\to h\Delta_{\mu\nu}h^\dagger,\\
\chi_\pm&=u^\dagger\chi u^\dagger\pm u\chi^\dagger u, & \chi_\pm&\to h\chi_\pm h^\dagger,
\end{aligned}$$

with the chiral connection

$$\Gamma_\mu=\frac{1}{2}(u^\dagger(\partial_\mu-\mathrm{i}r_\mu)u+u(\partial_\mu-\mathrm{i}l_\mu)u^\dagger), \quad \text{(C.1)}$$

where r_μ, l_μ are external right-/left-handed currents. As mentioned before, the general form of the Lagrangian up to order $\mathcal{O}(p^4)$ is

$$\mathcal{L}_{\rm HK\chi PT}=\mathcal{L}_\pi^{(2)}+\mathcal{L}_\pi^{(4)}+\mathcal{L}_{\pi KK}^{(1)}+\mathcal{L}_{\pi KK}^{(2)}+\mathcal{L}_{\pi KK}^{(3)}+\mathcal{L}_{\pi KK}^{(4)}, \quad \text{(C.2)}$$

where the purely pionic sector is chosen such that it coincides with the standard two flavor CHPT Lagrangian. Concerning the π–K interaction Lagrangian, it is clear from the discussion of the difficulties related to the power counting procedure in HKCHPT, see Appendix B, that there is not a one-to-one correspondence of the dimensions assigned to a Lagrangian term in the relativistic and the non-relativistic framework. This means that a given Lorentz invariant term can, via (B.5), give rise to contributions of different powers in the non-relativistic formulation. In Roessl's Lagrangian, the terms are labeled according to the leading non-relativistic contributions they lead to. The LECs are denoted A_i, B_i, C_i, and $M_{K,0}$ stands for the lowest order kaon mass. The HKCHPT Lagrangian thus reads

$$\mathcal{L}_{\pi KK}^{(1)}=D_\mu K^\dagger D^\mu K-M_{K,0}^2K^\dagger K, \quad \text{(C.3)}$$

$$\mathcal{L}_{\pi KK}^{(2)}=A_1\mathrm{Tr}(\Delta_\mu\Delta^\mu)K^\dagger K+A_2\mathrm{Tr}(\Delta^\mu\Delta^\nu)D_\mu K^\dagger D_\nu K+A_3K^\dagger\chi_+K+A_4\mathrm{Tr}(\chi_+)K^\dagger K, \quad \text{(C.4)}$$

$$\mathcal{L}_{\pi KK}^{(3)}=B_1(K^\dagger[\Delta^{\nu\mu},\Delta_\nu]D_\mu K-D_\mu K^\dagger[\Delta^{\nu\mu},\Delta_\nu]K)$$

$$
\begin{aligned}
&+ B_2 \mathrm{Tr}(\Delta^{\mu\nu}\Delta^{\rho})(D_{\mu\nu}K^\dagger D_\rho K + D_\rho K^\dagger D_{\mu\nu}K)\\
&+ B_3(K^\dagger[\Delta_\mu,\chi_-]D^\mu K - D_\mu K^\dagger[\Delta^\mu,\chi_-]K), \quad \text{(C.5)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}^{(4)}_{\pi KK} &= C_1 \mathrm{Tr}(\Delta_\nu\Delta^{\mu\nu})(K^\dagger D_\mu K + D_\mu K^\dagger K)\\
&+ C_2 \mathrm{Tr}(\Delta^{\mu\rho}\Delta^{\nu})(D_{\mu\nu}K^\dagger D_\rho K + D_\rho K^\dagger D_{\mu\nu}K)\\
&+ C_3\Big(\mathrm{Tr}(\Delta^{\mu\nu}\Delta^{\rho})(D_{\mu\nu}K^\dagger D_\rho K + D_\rho K^\dagger D_{\mu\nu}K)\\
&- 2(D^{\mu\nu}K^\dagger\Delta_\mu\Delta_{\nu\rho}D^\rho K + D^\rho K^\dagger \Delta_{\nu\rho}\Delta_\mu D^{\mu\nu}K)\Big)\\
&+ C_4 \mathrm{Tr}(\Delta^{\mu\nu}\Delta^{\rho\sigma})(D_{\mu\nu}K^\dagger D_{\rho\sigma}K + D_{\rho\sigma}K^\dagger D_{\mu\nu}K)\\
&+ C_5(D_\mu K^\dagger\chi_+ D^\mu K - M_K^2 K^\dagger\chi_+ K)\\
&+ C_6(\mathrm{Tr}(\chi_+)D_\mu K^\dagger D^\mu K - M_K^2\mathrm{Tr}(\chi_+)K^\dagger K)\\
&+ C_7 \mathrm{Tr}(\Delta_\mu\chi_-)(K^\dagger D_\mu K + D_\mu K^\dagger K)\\
&+ C_8 \mathrm{Tr}(\Delta_\mu\Delta^\mu)K^\dagger\chi_+ K\\
&+ C_9 \mathrm{Tr}(\Delta_\mu\Delta^\mu)\mathrm{Tr}(\chi_+)K^\dagger K\\
&+ C_{10}\mathrm{Tr}(\Delta^\mu\Delta^\nu)(D_\mu K^\dagger\chi_+ D_\nu K + D_\nu K^\dagger\chi_+ D_\mu K)\\
&+ C_{11}\mathrm{Tr}(\Delta^\mu\Delta^\nu)\mathrm{Tr}(\chi_+)(D_\mu K^\dagger D_\nu K + D_\nu K^\dagger D_\mu K)\\
&+ C_{12}D_\mu K^\dagger\{\{\Delta^\mu,\Delta^\nu\},\chi_+\}D_\nu K\\
&+ C_{13}\mathrm{Tr}(\chi_+)K^\dagger\chi_+ K + C_{14}\mathrm{Tr}(\chi_+^2)K^\dagger K\\
&+ C_{15}(\mathrm{Tr}(\chi_+))^2 K^\dagger K + C_{16}\mathrm{Tr}(\chi_-^2)K^\dagger K. \qquad \text{(C.6)}
\end{aligned}
$$

From the power counting formula (B.8) it follows that loops start contributing to amplitudes at third order. The infinities they generate are handled in the standard way, i.e. by renormalizing the LECs. Since $\mathcal{L}^{(2)}_{\pi KK}$ only accounts for second order tree contributions, the A_i are finite. However, for reasons of notational consistency, we write A_i^r in formulae describing renormalized observables, in analogy with B_i^r and C_i^r.

D Pion–kaon scattering amplitude in heavy-kaon CHPT

In this appendix, we present the pion–kaon scattering amplitude $T^{3/2}_{\pi K\to\pi K}(\nu,t)$. As noted before, it does not agree with the one given in [6] at various places. To one-loop accuracy, it takes the form

$$
\begin{aligned}
T^{3/2}_{\pi K\to\pi K}(\nu,t) &= -\frac{1}{4F_\pi^2}\nu - \frac{A_2^r}{16F_\pi^2}\nu^2 + \frac{A_1^r}{2F_\pi^2}t\\
&+ (-A_1^r - 2A_3^r - 4A_4^r)\frac{1}{F_\pi^2}M_\pi^2\\
&- \frac{C_3^r}{16F_\pi^2}\nu^3 - \frac{B_1^r}{4F_\pi^2}\nu t + \left(\frac{B_1^r}{2} - 2B_3^r\right)\frac{1}{F_\pi^2}\nu M_\pi^2\\
&+ \frac{1}{(4\pi)^2}\frac{1}{F_\pi^4}\left(-\frac{1}{36}\nu t + \frac{1}{6}\nu M_\pi^2\right) - \frac{C_4^r}{32F_\pi^2}\nu^4\\
&+ \left(-\frac{3B_2^r}{16} + \frac{C_2^r}{16} - \frac{C_3^r}{4}\right)\frac{1}{F_\pi^2}\nu^2 t + \left(-\frac{A_2^r}{16} + \frac{C_1^r}{4}\right)\frac{1}{F_\pi^2}t^2\\
&+ \left(\frac{A_2^r C_5^r}{8} + \frac{A_2^r C_6^r}{4} - \frac{C_{10}^r}{4} - \frac{C_{11}^r}{2} - \frac{C_{12}^r}{4}\right)\frac{1}{F_\pi^2}\nu^2 M_\pi^2\\
&+ \Big(-A_1^r C_5^r - 2A_1^r C_6^r - \frac{C_1^r}{2} + C_5^r\\
&+ 2C_6^r + 2C_7^r + C_8^r + 2C_9^r\Big)\frac{1}{F_\pi^2}tM_\pi^2\\
&+ \Big(2A_1^r C_5^r + 4A_1^r C_6^r + 4A_3^r C_5^r + 8A_3^r C_6^r + 8A_4^r C_5^r\\
&+ 16A_4^r C_6^r + \frac{4A_3^r l_3^r}{F_\pi^2} + \frac{8A_4^r l_3^r}{F_\pi^2} - 2C_8^r - 4C_9^r\\
&- 16C_{13}^r - 16C_{14}^r - 32C_{15}^r - 16C_{16}^r\Big)\frac{1}{F_\pi^2}M_\pi^4\\
&+ \frac{1}{(4\pi)^2}\frac{1}{F_\pi^4}\left(\frac{A_2^r M_K^2}{18}t^2 - \frac{13A_2^r M_K^2}{36}tM_\pi^2 + \frac{A_2^r M_K^2}{6}M_\pi^4\right)\\
&+ \Bigg(\left(\frac{1}{6}\nu M_\pi^2 + \left(-\frac{1}{8M_K^2} + \frac{A_2^r}{8}\right)\nu^2 M_\pi^2\right.\\
&+ \left(-\frac{1}{2} + \frac{A_2^r M_K^2}{6}\right)tM_\pi^2\\
&+ \left(1 + \frac{3A_1^r}{2} - \frac{A_2^r M_K^2}{12} + 4A_3^r + 8A_4^r\right)M_\pi^4\Bigg)\frac{1}{F_\pi^4}\mu_\pi\\
&+ \Bigg(\frac{1}{24}\nu t - \frac{1}{6}\nu M_\pi^2 + \left(-\frac{A_1^r}{2} - \frac{A_2^r M_K^2}{12}\right)t^2\\
&+ \left(\frac{5A_1^r}{4} + \frac{3A_2^r M_K^2}{8} + 2A_3^r + 4A_4^r\right)tM_\pi^2\\
&+ \left(-\frac{A_1^r}{2} - \frac{A_2^r M_K^2}{6} - A_3^r - 2A_4^r\right)M_\pi^4\Bigg)\frac{1}{F_\pi^4}J^r_{\pi\pi}(t) \qquad \text{(D.1)}\\
&+ \Bigg(-\frac{1}{32M_K}\nu^2 + \left(\frac{1}{64M_K^3} - \frac{A_1^r}{64M_K^3} - \frac{A_2^r}{64M_K}\right)\nu^3\\
&+ \frac{1}{16M_K}\nu t\\
&+ \left(-\frac{1}{8M_K} - \frac{A_3^r}{2M_K} - \frac{A_4^r}{M_K}\right)\nu M_\pi^2\Bigg)\frac{1}{F_\pi^4}J_\pi^r(x_-)\\
&+ \Bigg(\frac{3}{32M_K}\nu^2 + \left(\frac{3}{64M_K^3} + \frac{A_1^r}{64M_K^3} + \frac{A_2^r}{64M_K}\right)\nu^3\\
&+ \frac{3}{16M_K}\nu t + \left(-\frac{3}{8M_K} + \frac{A_3^r}{2M_K} + \frac{A_4^r}{M_K}\right)\nu M_\pi^2\Bigg)\frac{1}{F_\pi^4}J_\pi^r(x_+)\\
&+ \left(-\frac{1}{512M_K^4}\nu^4 + \frac{1}{32M_K^2}\nu^2 M_\pi^2\right)\frac{1}{F_\pi^4}G_\pi^r(x_-)\\
&+ \left(-\frac{3}{512M_K^4}\nu^4 + \frac{3}{32M_K^2}\nu^2 M_\pi^2\right)\frac{1}{F_\pi^4}G_\pi^r(x_+) + \mathcal{O}(p^6),
\end{aligned}
$$

in terms of the loop integrals

$$
\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2 - M_\pi^2}\frac{1}{\omega - v\cdot k} = 4\omega L + J_\pi^r(\omega), \qquad \text{(D.2)}
$$

$$
\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2 - M_\pi^2}\frac{1}{(\omega - v\cdot k)^2} = -4L + G_\pi^r(\omega), \qquad \text{(D.3)}
$$

with $L \sim 1/(d-4)$ as usual in dimensional regularization and $x_\pm = (\nu \pm t)/4M_K$. The loop function $J^r_{\pi\pi}(t)$ is taken from [8] and μ_π is defined in (3.2). Note also that in [6] matching relations were derived by comparing this amplitude to the one obtained in standard SU(3) CHPT [11]. Most of these are correct; however, in two cases we have found an error in the terms $\sim 1/\pi^2$ in the relations for B_1^r and B_3^r. The corrected matching conditions read

$$B_1^r = \frac{1}{F^2}\Bigg(-4L_3^r - \frac{5}{576\pi^2} - \frac{5}{108}\frac{\arctan(\sqrt{2})}{\sqrt{2}\pi^2} - \frac{31}{864}\frac{\ln(4/3)}{\pi^2} + \frac{1}{96}\frac{\ln(\bar{M}_K^2/\lambda^2)}{\pi^2} + \mathcal{O}(\bar{M}_K^2)\Bigg), \quad \text{(D.4)}$$

$$B_3^r = \frac{1}{F^2}\Bigg(-L_3^r + L_5^r + \frac{13}{2304\pi^2} - \frac{1}{108}\frac{\arctan(\sqrt{2})}{\sqrt{2}\pi^2} + \frac{7}{1728}\frac{\ln(4/3)}{\pi^2} - \frac{7}{768}\frac{\ln(\bar{M}_K^2/\lambda^2)}{\pi^2} + \mathcal{O}(\bar{M}_K^2)\Bigg). \quad \text{(D.5)}$$

The numerical analysis of the matching conditions derived from the πK amplitude leads to the numbers collected in Table 2.

E Form factors and S-wave projected scattering amplitudes

In this appendix, we collect the one-loop representations of the various non-strange scalar form factors and S-wave projected scattering amplitudes appearing in (5.4). The derivation of the scalar form factors Γ_π and Γ_η is completely analogous to the one of Γ_K. One finds

$$\begin{aligned}\Gamma_\pi(s) = {} & M_\pi^2 + \frac{M_\pi^2}{F^2}\Bigg[L_4^r(-16M_\pi^2+8s) \\ & + L_5^r(-8M_\pi^2+4s) + L_6^r 32M_\pi^2 + L_8^r 16M_\pi^2 \\ & + \left(\frac{1}{2}M_\pi^2 - s\right)J_{\pi\pi}^r(s) - \frac{1}{4}sJ_{KK}^r(s) - \frac{1}{18}M_\pi^2 J_{\eta\eta}^r(s)\Bigg] \\ & + \mathcal{O}(p^6), \qquad \text{(E.1)}\end{aligned}$$

$$\begin{aligned}\Gamma_\eta(s) = {} & \frac{M_\pi^2}{3} + \frac{M_\pi^2}{F^2}\Bigg[L_4^r\left(\frac{16}{3}M_\pi^2 - \frac{64}{3}M_K^2 + 8s\right) \\ & + L_5^r\left(\frac{40}{9}M_\pi^2 - \frac{64}{9}M_K^2 + \frac{4}{3}s\right) \\ & + L_6^r\left(-\frac{32}{3}M_\pi^2 + \frac{128}{3}M_K^2\right) + L_7^r\left(\frac{128}{3}M_\pi^2 - \frac{128}{3}M_K^2\right) \\ & + L_8^r\frac{16}{3}M_\pi^2 - \frac{2}{3}M_\pi^2\mu_\pi + \frac{2}{3}M_K^2\mu_K \\ & - \frac{1}{2}M_\pi^2 J_{\pi\pi}^r(s) + \left(\frac{2}{3}M_K^2 - \frac{3}{4}s\right)J_{KK}^r(s) \\ & + \left(\frac{7}{54}M_\pi^2 - \frac{8}{27}M_K^2\right)J_{\eta\eta}^r(s)\Bigg] + \mathcal{O}(p^6), \qquad \text{(E.2)}\end{aligned}$$

in terms of the physical meson masses and we set $F = F_\pi$ throughout. Next, we consider the various isospin zero $KK \to 2$ Goldstone bosons scattering amplitudes. From these, we consider the projection on the $l = 0$ components (S-waves) using (for generic Goldstone bosons Φ_a)

$$t^0_{0,\Phi_a\Phi_c\to\Phi_b\Phi_d}(s) = \frac{1}{64\pi}\int_{-1}^{1}\mathrm{d}z T^0_{\Phi_a\Phi_c\to\Phi_b\Phi_d}(s,t,u), \quad \text{(E.3)}$$

with $z = \cos(\theta)$ and the angular dependence is implicitly contained in t and u. The pertinent Mandelstam variables are defined by the following kinematics:

$$K(p_1) + K(p_3) \to \Phi_a(p_2) + \Phi_a(p_4), \qquad \text{(E.4)}$$

where Φ_a stands for pion, kaon, or eta degrees of freedom. The coordinate frame is chosen such that θ is the angle included by $\vec{p}_1$ and $\vec{p}_2$. The results for the three amplitudes pertinent to our case will now be given.

$KK \to \pi\pi$ scattering

The Mandelstam variables for this configuration are

$$\begin{aligned} s &= 4(\vec{p}_1^{\,2} + M_K^2), \qquad \text{(E.5)} \\ t &= -2\vec{p}_1^{\,2} + M_\pi^2 - M_K^2 + 2|\vec{p}_1|\sqrt{\vec{p}_1^{\,2} - M_\pi^2 + M_K^2}\,z, \\ u &= -2\vec{p}_1^{\,2} + M_\pi^2 - M_K^2 - 2|\vec{p}_1|\sqrt{\vec{p}_1^{\,2} - M_\pi^2 + M_K^2}\,z. \end{aligned}$$

The $I = 0, l = 0$ partial amplitude is then given by the following expression:

$$\begin{aligned} t^0_{0,KK\to\pi\pi}(s) = {} & -\frac{1}{64\pi}\sqrt{\frac{3}{2}} \\ & \Bigg\{\frac{s}{F_\pi^2} + \frac{1}{F^4}\big(L_1^r(128M_\pi^2M_K^2 - 64M_\pi^2 s - 64M_K^2 s + 32s^2) \\ & + L_2^r\left(\frac{128}{3}M_\pi^2M_K^2 - \frac{32}{3}M_\pi^2 s - \frac{32}{3}M_K^2 s + \frac{32}{3}s^2\right) \\ & + L_3^r\left(\frac{128}{3}M_\pi^2M_K^2 - \frac{56}{3}M_\pi^2 s - \frac{56}{3}M_K^2 s + \frac{32}{3}s^2\right) \\ & + L_4^r(-128M_\pi^2M_K^2 + 32M_\pi^2 s + 32M_K^2 s) \\ & + L_5^r(-32M_\pi^2M_K^2 + 8M_\pi^2 s) \\ & + L_6^r 128M_\pi^2M_K^2 + L_8^r 64M_\pi^2M_K^2 \\ & + \Bigg(\frac{1}{2}M_\pi^4 - \frac{13}{6}M_\pi^2M_K^2 - \frac{1}{2}M_K^4 + \frac{19}{24}M_\pi^2 s \\ & + \frac{7}{24}M_K^2 s - \frac{65}{48}s^2\Bigg)\mu_\pi \\ & + \Bigg(-\frac{11}{27}M_\pi^4 - \frac{8}{3}M_\pi^2M_K^2 + \frac{20}{27}M_K^4 + \frac{1}{4}M_\pi^2 s \\ & + \frac{5}{4}M_K^2 s - \frac{9}{8}s^2\Bigg)\mu_K \\ & + \Bigg(-\frac{5}{54}M_\pi^4 + \frac{11}{18}M_\pi^2M_K^2 - \frac{13}{54}M_K^4 - \frac{13}{24}M_\pi^2 s \\ & + \frac{11}{24}M_K^2 s - \frac{1}{48}s^2\Bigg)\mu_\eta + \left(\frac{1}{2}M_\pi^2 s - s^2\right)\tilde{J}_{\pi\pi}^r(s) \\ & - \frac{3}{4}s^2\tilde{J}_{KK}^r(s) + \left(\frac{4}{9}M_\pi^2M_K^2 - \frac{1}{2}M_\pi^2 s\right)\tilde{J}_{\eta\eta}^r(s) \\ & + \frac{1}{(4\pi)^2}\Bigg(-\frac{1}{18}M_\pi^4 + \frac{59}{9}M_\pi^2M_K^2 + \frac{23}{18}M_K^4 \\ & - \frac{53}{36}M_\pi^2 s - \frac{125}{36}M_K^2 s + \frac{43}{18}s^2\Bigg) \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\sqrt{s-4M_\pi^2}\sqrt{s-4M_K^2}} \\
&\times \ln\left(\frac{2M_\pi^2+2M_K^2-s-\sqrt{s-4M_\pi^2}\sqrt{s-4M_K^2}}{2M_\pi^2+2M_K^2-s+\sqrt{s-4M_\pi^2}\sqrt{s-4M_K^2}}\right) \\
&\times \Bigg(\Bigg(\frac{1}{2}M_\pi^6 + 2M_\pi^4M_K^2 - 2M_\pi^2M_K^4 - \frac{1}{2}M_K^6 - \frac{3}{8}M_\pi^4 s \\
&+ \frac{3}{8}M_K^4 s\Bigg)\mu_\pi + \Bigg(-\frac{11}{27}M_\pi^6 - \frac{7}{3}M_\pi^4M_K^2 + 2M_\pi^2M_K^4 \\
&+ \frac{20}{27}M_K^6 + \frac{5}{12}M_\pi^4 s - \frac{1}{3}M_\pi^2M_K^2 s - \frac{1}{12}M_K^4 s\Bigg)\mu_K \\
&+ \Bigg(-\frac{5}{54}M_\pi^6 + \frac{1}{3}M_\pi^4M_K^2 - \frac{13}{54}M_K^6 - \frac{1}{24}M_\pi^4 s \\
&+ \frac{1}{3}M_\pi^2M_K^2 s - \frac{7}{24}M_K^4 s\Bigg)\mu_\eta \\
&+ \frac{1}{(4\pi)^2}\Bigg(-\frac{1}{18}M_\pi^6 + \frac{25}{18}M_\pi^4M_K^2 - \frac{47}{18}M_\pi^2M_K^4 \\
&+ \frac{23}{18}M_K^6 - \frac{5}{18}M_\pi^4 s + \frac{5}{9}M_\pi^2M_K^2 s - \frac{5}{18}M_K^4 s\Bigg)\Bigg) \\
&+ \int_{-1}^{1}\mathrm{d}z\,\Bigg(\Bigg(-\frac{1}{8}\frac{M_\pi^8}{t^2} - \frac{1}{4}M_\pi^4 + \frac{1}{2}\frac{M_\pi^6M_K^2}{t^2} - \frac{3}{4}\frac{M_\pi^4M_K^4}{t^2} \\
&- \frac{3}{2}M_\pi^2M_K^2 + \frac{1}{2}\frac{M_\pi^2M_K^6}{t^2} - \frac{1}{8}\frac{M_K^8}{t^2} - \frac{1}{4}M_K^4 - \frac{1}{16}\frac{M_\pi^4 s}{t} \\
&+ \frac{1}{8}M_\pi^2 s + \frac{1}{8}\frac{M_\pi^2M_K^2 s}{t} - \frac{1}{16}\frac{M_K^4 s}{t} \\
&+ \frac{1}{8}M_K^2 s - \frac{1}{16}st + M_\pi^2 t + M_K^2 t - \frac{5}{8}t^2\Bigg)\tilde{J}^r_{\pi K}(t) \\
&+ \Bigg(-\frac{1}{72}\frac{M_\pi^8}{t^2} - \frac{1}{18}\frac{M_\pi^6}{t} + \frac{1}{36}M_\pi^4 + \frac{1}{18}\frac{M_\pi^6M_K^2}{t^2} \\
&+ \frac{2}{9}\frac{M_\pi^4M_K^2}{t} - \frac{1}{12}\frac{M_\pi^4M_K^4}{t^2} - \frac{7}{18}M_\pi^2M_K^2 - \frac{5}{18}\frac{M_\pi^2M_K^4}{t} \\
&+ \frac{1}{18}\frac{M_\pi^2M_K^6}{t^2} - \frac{1}{72}\frac{M_K^8}{t^2} + \frac{1}{9}\frac{M_K^6}{t} - \frac{11}{36}M_K^4 \\
&- \frac{1}{144}\frac{M_\pi^4 s}{t} - \frac{1}{24}M_\pi^2 s + \frac{1}{72}\frac{M_\pi^2M_K^2 s}{t} - \frac{1}{144}\frac{M_K^4 s}{t} \\
&+ \frac{7}{24}M_K^2 s - \frac{1}{16}st + \frac{1}{6}M_\pi^2 t + \frac{1}{3}M_K^2 t - \frac{1}{8}t^2\Bigg)\tilde{J}^r_{K\eta}(t) \\
&+ \Bigg(-\frac{1}{8}\frac{M_\pi^8}{u^2} - \frac{3}{4}M_\pi^4 + \frac{1}{2}\frac{M_\pi^6M_K^2}{u^2} \\
&- \frac{3}{4}\frac{M_\pi^4M_K^4}{u^2} - \frac{5}{2}M_\pi^2M_K^2 \\
&+ \frac{1}{2}\frac{M_\pi^2M_K^6}{u^2} - \frac{1}{8}\frac{M_K^8}{u^2} - \frac{3}{4}M_K^4 - \frac{1}{16}\frac{M_\pi^4 s}{u} + \frac{3}{2}M_\pi^2 s \\
&+ \frac{1}{8}\frac{M_\pi^2M_K^2 s}{u} - \frac{1}{16}\frac{M_K^4 s}{u} + \frac{3}{2}M_K^2 s \\
&- \frac{9}{16}s^2 - \frac{19}{16}st + \frac{3}{2}M_\pi^2 t + \frac{3}{2}M_K^2 t - \frac{5}{8}t^2\Bigg)\tilde{J}^r_{\pi K}(u) \\
&+ \Bigg(-\frac{1}{72}\frac{M_\pi^8}{u^2} - \frac{1}{18}\frac{M_\pi^6}{u} - \frac{5}{36}M_\pi^4 + \frac{1}{18}\frac{M_\pi^6M_K^2}{u^2} \\
&+ \frac{2}{9}\frac{M_\pi^4M_K^2}{u} - \frac{1}{12}\frac{M_\pi^4M_K^4}{u^2} - \frac{7}{18}M_\pi^2M_K^2 - \frac{5}{18}\frac{M_\pi^2M_K^4}{u} \\
&+ \frac{1}{18}\frac{M_\pi^2M_K^6}{u^2} - \frac{1}{72}\frac{M_K^8}{u^2} + \frac{1}{9}\frac{M_K^6}{u} - \frac{5}{36}M_K^4 - \frac{1}{144}\frac{M_\pi^4 s}{u} \\
&+ \frac{1}{6}M_\pi^2 s + \frac{1}{72}\frac{M_\pi^2M_K^2 s}{u} - \frac{1}{144}\frac{M_K^4 s}{u} + \frac{1}{3}M_K^2 s - \frac{1}{16}s^2 \\
&- \frac{3}{16}st + \frac{1}{3}M_\pi^2 t + \frac{1}{6}M_K^2 t - \frac{1}{8}t^2\Bigg)\tilde{J}^r_{K\eta}(u)\Bigg)\Bigg)\Bigg\} \\
&+ \mathcal{O}(p^6). \qquad \text{(E.6)}
\end{aligned}
$$

$KK \to KK$ scattering

The Mandelstam variables for this configuration are

$$
s = 4(\vec{p}_1{}^2 + M_K^2), \quad t = 2\vec{p}_1{}^2(z-1), \quad u = -2\vec{p}_1{}^2(z+1). \qquad \text{(E.7)}
$$

In this case, besides the t-channel and the u-channel loop functions, the terms proportional to $1/t$ cannot be integrated analytically. The $I=0, l=0$ partial amplitude is then given by the following expression:

$$
\begin{aligned}
&t^0_{0,KK\to KK}(s) = \frac{1}{64\pi} \\
&\times \Bigg\{\frac{3s}{2F_\pi^2} + \frac{1}{F^4}\Bigg(L_1^r\left(\frac{448}{3}M_K^4 - \frac{416}{3}M_K^2 s + \frac{112}{3}s^2\right) \\
&+ L_2^r\left(\frac{256}{3}M_K^4 - \frac{176}{3}M_K^2 s + \frac{64}{3}s^2\right) \\
&+ L_3^r(64M_K^4 - 56M_K^2 s + 16s^2) \\
&+ L_4^r(-128M_K^4 + 48M_K^2 s) \\
&+ L_5^r(-48M_K^4 + 12M_\pi^2 s) + L_6^r 192M_K^4 + L_8^r 96M_K^4 \\
&+ \left(\frac{1}{2}M_\pi^2M_K^2 - \frac{13}{6}M_K^4 - \frac{3}{2}M_\pi^2 s + \frac{11}{24}M_K^2 s - \frac{13}{24}s^2\right)\mu_\pi \\
&+ \left(-\frac{16}{3}M_K^4 + \frac{35}{12}M_K^2 s - \frac{41}{24}s^2\right)\mu_K \\
&+ \left(-\frac{1}{2}M_\pi^2M_K^2 - \frac{17}{6}M_K^4 - \frac{3}{4}M_\pi^2 s + \frac{45}{8}M_K^2 s - \frac{3}{2}s^2\right)\mu_\eta \\
&- \frac{3}{8}s^2\tilde{J}^r_{\pi\pi}(s) - \frac{9}{8}s^2\tilde{J}^r_{KK}(s) \\
&+ \left(-\frac{8}{9}M_K^4 + 2M_K^2 s - \frac{9}{8}s^2\right)\tilde{J}^r_{\eta\eta}(s) + \frac{1}{(4\pi)^2} \\
&\times \left(2M_\pi^2M_K^2 + \frac{41}{3}M_K^4 - \frac{3}{2}M_\pi^2 s - \frac{107}{12}M_K^2 s + \frac{43}{12}s^2\right) \\
&+ \int_{-1}^{1}\mathrm{d}z\,\Bigg(\Bigg(-\frac{2}{3}\frac{M_\pi^2M_K^4}{t} + \frac{2}{3}\frac{M_K^6}{t}\Bigg)\mu_\pi \\
&+ \left(\frac{2}{3}\frac{M_\pi^2M_K^4}{t} - \frac{2}{3}\frac{M_K^6}{t}\right)\mu_\eta + \Bigg(-M_\pi^2M_K^2 + \frac{1}{2}M_\pi^2 s \\
&- \frac{1}{8}st + \frac{1}{4}M_\pi^2 t + \frac{1}{4}M_K^2 t - \frac{5}{32}t^2\Bigg)\tilde{J}^r_{\pi\pi}(t) \\
&+ \left(-2M_K^4 + M_K^2 s - \frac{1}{4}st + M_K^2 t - \frac{1}{2}t^2\right)\tilde{J}^r_{KK}(t) \\
&+ \left(-\frac{2}{9}M_K^4 + \frac{1}{2}M_K^2 t - \frac{9}{32}t^2\right)\tilde{J}^r_{\eta\eta}(t)
\end{aligned}
$$

$$+ \left(-M_K^4 + \frac{3}{2}M_K^2 t - \frac{9}{16}t^2\right)\tilde{J}^r_{\pi\eta}(t)$$
$$+ \left(-3M_K^4 + 3M_K^2 s - \frac{3}{4}s^2 - \frac{3}{2}st\right.$$
$$\left.\left.\left. + \; 3M_K^2 t - \frac{3}{4}t^2\right)\tilde{J}^r_{KK}(u)\right)\right)\Bigg\} + \mathcal{O}(p^6). \qquad \text{(E.8)}$$

$KK \to \eta\eta$ scattering

The Mandelstam variables for this case are

$$s = 4(\vec{p}_1{}^2 + M_K^2),$$
$$t = -2\vec{p}_1{}^2 - \frac{1}{3}M_\pi^2 + \frac{1}{3}M_K^2 + 2|\vec{p}_1|\sqrt{\vec{p}_1{}^2 + \frac{1}{3}M_\pi^2 - \frac{1}{3}M_K^2}\,z,$$
$$u = -2\vec{p}_1{}^2 - \frac{1}{3}M_\pi^2 + \frac{1}{3}M_K^2 - 2|\vec{p}_1|\sqrt{\vec{p}_1{}^2 + \frac{1}{3}M_\pi^2 - \frac{1}{3}M_K^2}\,z. \qquad \text{(E.9)}$$

The $I = 0, l = 0$ partial amplitude is then given by the following expression:

$$t^0_{0,KK\to\eta\eta}(s) = \frac{1}{64\pi}\sqrt{2}\left\{\frac{1}{2F_\pi^2}\left(-\frac{4}{3}M_K^2 + \frac{3}{2}s\right)\right.$$
$$+ \frac{1}{F^4}\left(L_1^r\left(-\frac{64}{3}M_\pi^2 M_K^2 + \frac{256}{3}M_K^4 + \frac{32}{3}M_\pi^2 s\right.\right.$$
$$\left. - \frac{224}{3}M_K^2 s + 16s^2\right) + L_2^r\left(-\frac{64}{9}M_\pi^2 M_K^2 + \frac{256}{9}M_K^4\right.$$
$$\left. + \frac{16}{9}M_\pi^2 s - \frac{112}{9}M_K^2 s + \frac{16}{3}s^2\right) + L_3^r\left(-\frac{256}{27}M_\pi^2 M_K^2\right.$$
$$\left. + \frac{1024}{27}M_K^4 + \frac{124}{27}M_\pi^2 s - \frac{868}{27}M_K^2 s + \frac{64}{9}s^2\right)$$
$$+ L_4^r\left(\frac{64}{3}M_\pi^2 M_K^2 - \frac{256}{3}M_K^4 - \frac{16}{3}M_\pi^2 s + \frac{112}{3}M_K^2 s\right)$$
$$+ L_5^r\left(-\frac{16}{9}M_\pi^2 M_K^2 - \frac{224}{9}M_K^4 + 12M_\pi^2 s\right)$$
$$+ L_6^r\left(-\frac{64}{3}M_\pi^2 M_K^2 + \frac{256}{3}M_K^4\right)$$
$$+ L_7^r\left(-\frac{128}{3}M_\pi^2 M_K^2 + \frac{128}{3}M_K^4\right)$$
$$+ L_8^r(-32M_\pi^2 M_K^2 + 64M_K^4)$$
$$+ \left(\frac{3}{4}M_\pi^4 - \frac{17}{12}M_\pi^2 M_K^2 + \frac{37}{12}M_K^4 - \frac{155}{48}M_\pi^2 s + \frac{29}{48}M_K^2 s\right.$$
$$\left. - \frac{1}{32}s^2\right)\mu_\pi + \left(-\frac{7}{6}M_\pi^4 + \frac{23}{3}M_\pi^2 M_K^2 - \frac{86}{9}M_K^4\right.$$
$$\left. + \frac{1}{24}M_\pi^2 s + \frac{65}{24}M_K^2 s - \frac{19}{16}s^2\right)\mu_K$$
$$+ \left(\frac{5}{12}M_\pi^4 - \frac{335}{108}M_\pi^2 M_K^2\right.$$
$$\left. + \frac{347}{108}M_K^4 + \frac{29}{48}M_\pi^2 s - \frac{47}{48}M_K^2 s - \frac{1}{32}s^2\right)\mu_\eta$$
$$- \frac{1}{4}M_\pi^2 s\tilde{J}^r_{\pi\pi}(s) + \left(M_K^2 s - \frac{9}{8}s^2\right)\tilde{J}^r_{KK}(s)$$
$$+ \left(-\frac{14}{27}M_\pi^2 M_K^2 + \frac{32}{27}M_K^4 + \frac{7}{12}M_\pi^2 s - \frac{4}{3}M_K^2 s\right)\tilde{J}^r_{\eta\eta}(s)$$
$$+ \frac{1}{(4\pi)^2}\left(\frac{5}{4}M_\pi^4 - \frac{149}{54}M_\pi^2 M_K^2 + \frac{751}{108}M_K^4 - \frac{67}{72}M_\pi^2 s\right.$$
$$\left. - \frac{179}{72}M_K^2 s + \frac{13}{12}s^2\right)$$
$$+ \frac{1}{\sqrt{3}\sqrt{3s + 4M_\pi^2 - 16M_K^2}\sqrt{s - 4M_K^2}}$$
$$\times \ln\left(\left(-2M_\pi^2 + 14M_K^2 - 3s\right.\right.$$
$$\left. - \sqrt{3}\sqrt{3s + 4M_\pi^2 - 16M_K^2}\sqrt{s - 4M_K^2}\right)$$
$$\Big/ \left(-2M_\pi^2 + 14M_K^2 - 3s\right.$$
$$\left.\left. + \sqrt{3}\sqrt{3s + 4M_\pi^2 - 16M_K^2}\sqrt{s - 4M_K^2}\right)\right)$$
$$\times \left(\left(-\frac{3}{4}M_\pi^6 + \frac{9}{2}M_\pi^4 M_K^2 - \frac{15}{4}M_K^6 - \frac{27}{16}M_\pi^4 s\right.\right.$$
$$\left. + \frac{27}{16}M_K^4 s\right)\mu_\pi + \left(\frac{7}{6}M_\pi^6 - \frac{17}{2}M_\pi^4 M_K^2 + \frac{116}{9}M_\pi^2 M_K^4\right.$$
$$\left. - \frac{50}{9}M_K^6 + \frac{15}{8}M_\pi^4 s - \frac{3}{2}M_\pi^2 M_K^2 s - \frac{3}{8}M_K^4 s\right)\mu_K$$
$$+ \left(-\frac{5}{12}M_\pi^6 + 4M_\pi^4 M_K^2 - \frac{116}{9}M_\pi^2 M_K^4 + \frac{335}{36}M_K^6\right.$$
$$\left. - \frac{3}{16}M_\pi^4 s + \frac{3}{2}M_\pi^2 M_K^2 s - \frac{21}{16}M_K^4 s\right)\mu_\eta$$
$$+ \frac{1}{(4\pi)^2}\left(-\frac{5}{4}M_\pi^6 + \frac{79}{12}M_\pi^4 M_K^2 - \frac{113}{12}M_\pi^2 M_K^4\right.$$
$$\left.\left. + \frac{49}{12}M_K^6 - \frac{5}{4}M_\pi^4 s + \frac{5}{2}M_\pi^2 M_K^2 s - \frac{5}{4}M_K^4 s\right)\right)$$
$$+ \int_{-1}^{1}\mathrm{d}z\left(\left(-\frac{1}{48}\frac{M_\pi^8}{t^2} - \frac{1}{12}\frac{M_\pi^6}{t} + \frac{1}{24}M_\pi^4 + \frac{1}{12}\frac{M_\pi^6 M_K^2}{t^2}\right.\right.$$
$$+ \frac{1}{3}\frac{M_\pi^4 M_K^2}{t} - \frac{1}{8}\frac{M_\pi^4 M_K^4}{t^2} - \frac{7}{12}M_\pi^2 M_K^2 - \frac{5}{12}\frac{M_\pi^2 M_K^4}{t}$$
$$+ \frac{1}{12}\frac{M_\pi^2 M_K^6}{t^2} - \frac{1}{48}\frac{M_K^8}{t^2} + \frac{1}{6}\frac{M_K^6}{t} - \frac{11}{24}M_K^4$$
$$- \frac{3}{32}\frac{M_\pi^4 s}{t} + \frac{3}{16}M_\pi^2 s + \frac{3}{16}\frac{M_\pi^2 M_K^2 s}{t} + \frac{3}{16}M_K^2 s$$
$$\left. - \frac{3}{32}\frac{M_K^4 s}{t} - \frac{3}{32}st + \frac{1}{4}M_\pi^2 t + \frac{1}{2}M_K^2 t - \frac{3}{16}t^2\right)\tilde{J}^r_{\pi,K}(t)$$
$$+ \left(-\frac{1}{432}\frac{M_\pi^8}{t^2} - \frac{1}{36}\frac{M_\pi^6}{t} - \frac{1}{8}M_\pi^4 + \frac{1}{108}\frac{M_\pi^6 M_K^2}{t^2}\right.$$

$$
\begin{aligned}
&+ \frac{7}{36}\frac{M_\pi^4 M_K^2}{t} - \frac{1}{72}\frac{M_\pi^4 M_K^4}{t^2} + \frac{11}{12}M_\pi^2 M_K^2 - \frac{11}{36}\frac{M_\pi^2 M_K^4}{t} \\
&+ \frac{1}{108}\frac{M_\pi^2 M_K^6}{t^2} - \frac{1}{432}\frac{M_K^8}{t^2} + \frac{5}{36}\frac{M_K^6}{t} - \frac{161}{72}M_K^4 \\
&- \frac{1}{96}\frac{M_\pi^4 s}{t} - \frac{1}{16}M_\pi^2 s + \frac{1}{48}\frac{M_\pi^2 M_K^2 s}{t} + \frac{7}{16}M_K^2 s \\
&- \frac{1}{96}\frac{M_K^4 s}{t} - \frac{3}{32}st - \frac{1}{4}M_\pi^2 t + \frac{5}{4}M_K^2 t - \frac{3}{16}t^2\Big)\tilde{J}^r_{K\eta}(t) \\
&+ \Big(-\frac{1}{48}\frac{M_\pi^8}{u^2} - \frac{1}{12}\frac{M_\pi^6}{u} - \frac{5}{24}M_\pi^4 + \frac{1}{12}\frac{M_\pi^6 M_K^2}{u^2} \\
&+ \frac{1}{3}\frac{M_\pi^4 M_K^2}{u} - \frac{1}{8}\frac{M_\pi^4 M_K^4}{u^2} + \frac{17}{12}M_\pi^2 M_K^2 - \frac{5}{12}\frac{M_\pi^2 M_K^4}{u} \\
&+ \frac{1}{12}\frac{M_\pi^2 M_K^6}{u^2} - \frac{1}{48}\frac{M_K^8}{u^2} + \frac{1}{6}\frac{M_K^6}{u} - \frac{53}{24}M_K^4 - \frac{3}{32}\frac{M_\pi^4 s}{u} \\
&- \frac{1}{4}M_\pi^2 s + \frac{3}{16}\frac{M_\pi^2 M_K^2 s}{u} + M_K^2 s - \frac{3}{32}\frac{M_K^4 s}{u} \\
&- \frac{9}{32}st - \frac{1}{2}M_\pi^2 t + \frac{5}{4}M_K^2 t - \frac{3}{16}t^2\Big)\tilde{J}^r_{\pi,K}(u) \\
&+ \Big(-\frac{1}{432}\frac{M_\pi^8}{u^2} - \frac{1}{36}\frac{M_\pi^6}{u} - \frac{1}{24}M_\pi^4 + \frac{1}{108}\frac{M_\pi^6 M_K^2}{u^2} \\
&+ \frac{7}{36}\frac{M_\pi^4 M_K^2}{u} - \frac{1}{72}\frac{M_\pi^4 M_K^4}{u^2} + \frac{1}{12}M_\pi^2 M_K^2 \\
&- \frac{11}{36}\frac{M_\pi^2 M_K^4}{u} + \frac{1}{108}\frac{M_\pi^2 M_K^6}{u^2} - \frac{1}{432}\frac{M_K^8}{u^2} + \frac{5}{36}\frac{M_K^6}{u} \\
&- \frac{35}{72}M_K^4 - \frac{1}{96}\frac{M_\pi^4 s}{u} + \frac{1}{48}\frac{M_\pi^2 M_K^2 s}{u} \\
&+ \frac{1}{2}M_K^2 s - \frac{1}{96}\frac{M_K^4 s}{u} - \frac{3}{32}s^2 - \frac{9}{32}st \\
&+ \frac{1}{2}M_K^2 t - \frac{3}{16}t^2\Big)\tilde{J}^r_{K\eta}(u)\Big)\Big)\Big\} + \mathcal{O}(p^6). \qquad \text{(E.10)}
\end{aligned}
$$

Throughout, we have employed the modified loop functions

$$\tilde{J}_{aa}(t) = J_{aa}(t) + \frac{1}{(4\pi)^2} - \mu_a, \qquad \text{(E.11)}$$

$$
\begin{aligned}
\tilde{J}_{ab}(t) &= J_{ab}(t) + \frac{1}{(4\pi)^2} + \frac{M_a^2 - M_b^2}{2t}(\mu_b - \mu_a) \\
&\quad - \frac{1}{2}(\mu_b + \mu_a), \qquad \text{(E.12)}
\end{aligned}
$$

for $a, b \in \{\pi, K, \eta\}$.

References

1. L.S. Brown, W.J. Pardee, R.D. Peccei, Phys. Rev. D **4**, 2801 (1971)
2. T.P. Cheng, R. Dashen, Phys. Rev. Lett. **26**, 594 (1971)
3. J. Gasser, M.E. Sainio, in Physics and Detectors for DAΦNE, Frascati Physics Series Vol. 16, edited by S. Bianco et al. (Frascati, 1999) [hep-ph/0002283]
4. B. Ananthanarayan, P. Büttiker, Eur. Phys. J. C **19**, (2001) [hep-ph/0012023]
5. B. Adeva et al., CERN/SPSC 2000-032
6. A. Roessl, Nucl. Phys. B **555**, 507 (1999) [hep-ph/9904230]
7. S.M. Ouellette, hep-ph/0101055
8. J. Gasser, H. Leutwyler, Nucl. Phys. B **250**, 465 (1985)
9. J. Gasser, Ulf-G. Meißner, Nucl. Phys. B **357**, 90 (1991)
10. S. Bellucci, J. Gasser, M.E. Sainio, Nucl. Phys. B **423**, 80 (1994) [Erratum-ibid. B **431**, 413 (1994)] [hep-ph/9401206]; E. Golowich, J. Kambor, Nucl. Phys. B **447**, 373 (1995) [hep-ph/9501318]; J. Bijnens et al., Phys. Lett. B **374**, 210 (1996) [hep-ph/9511397]; H.W. Fearing, S. Scherer, Phys. Rev. D **53**, 315 (1996) [hep-ph/9408346]; P. Post, K. Schilcher, Phys. Rev. Lett. **79**, 4088 (1997) [hep-ph/9701422]; J. Bijnens, G. Colangelo, G. Ecker, JHEP **9902**, 020 (1999) [hep-ph/9902437]; J. Bijnens, G. Colangelo, G. Ecker, Ann. Phys. **280**, 100 (2000) [hep-ph/9907333]
11. V. Bernard, N. Kaiser, Ulf-G. Meißner, Nucl. Phys. B **357**, 129 (1991)
12. M. Knecht et al., Phys. Lett. B **313**, 229 (1993) [hep-ph/9305332]
13. Ulf-G. Meißner, J.A. Oller, Nucl. Phys. A **679**, 671 (2001) [hep-ph/0005253]
14. J. Bijnens, G. Ecker, J. Gasser, in The second DAΦNE Physics Handbook, edited by L. Maiani, G. Pancheri, N. Paver (INFN, Frascati) (1995) [hep-ph/9411232]
15. G. Amoros, J. Bijnens, P. Talavera, Nucl. Phys. B **602**, 87 (2001) [hep-ph/0101127]
16. C.G. Callan, I.R. Klebanov, Nucl. Phys. B **262**, 365 (1985)
17. B. Ananthanarayan, P. Büttiker, B. Moussallam, Eur. Phys. J. C **22**, (2001) [hep-ph/0106230]
18. M. Jamin, J.A. Oller, A. Pich, Nucl. Phys. B **587**, 331 (2000) [hep-ph/0006045]
19. E. Jenkins, A.V. Manohar, Phys. Lett. B **255**, 558 (1991)
20. V. Bernard, N. Kaiser, J. Kambor, Ulf-G. Meißner, Nucl. Phys. B **388**, 315 (1992)
21. S.R. Coleman, J. Wess, B. Zumino, Phys. Rev. **177**, 2239 (1969); C.G. Callan, S.R. Coleman, J. Wess, B. Zumino, Phys. Rev. **177**, 2247 (1969)
22. J. Gasser, H. Leutwyler, Ann. Phys. (NY) **158**, 142 (1984)